BOOK ONE

General Mathematics

A PROBLEM SOLVING APPROACH

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Preface

GENERAL MATHEMATICS, A PROBLEM SOLVING APPROACH, Book One is the first of a two-book series designed to meet the needs of students enrolled in general mathematics courses. This series provides students with a foundation for a continuing study of mathematics and at the same time provides the terminal student with the mathematics needed for meaningful participation in the adult community.

This book is a revision and refinement of HOLT GENERAL MATHEMATICS. Many fine features of the former edition have been retained. New organization provides a clear picture of mathematics and its importance to the student. The modern point of view is utilized without losing sight of the vast wealth of material that has stood the test of classroom use over the years. Ideas, vocabulary, and concepts are presented in accord with the findings of recent experimentation in school mathematics.

Experience with mathematical thinking and knowledge of the nature of mathematics furnish a valuable frame of reference for living in a world of continual and rapid change. In this regard, the problem-solving approach challenges each student to do his best. Through the study of problems, each student can improve his computational skill; he can improve his reading ability; he can sharpen his reasoning power. The Problem-Solving Steps used in this series provide a logical procedure that can apply to a wide variety of non-mathematical situations. The realistic problem settings are designed for maximum student involvement.

The student who completes this book will have increased his competence to deal with mathematical situations, he will have seen the importance of mathematics in a wide variety of situations, and he will have acquired a mathematical background that will equip him to meet the demands of those situations in our modern society.

In presenting this book, the authors hope that the student will find increased confidence and interest in one of the most vital areas of the curriculum – mathematics. The authors wish to express their appreciation to the many teachers who have offered suggestions and constructive criticisms of previous editions. Their comments and suggestions have been an invaluable aid to the authors in writing GENERAL MATHEMATICS, A PROBLEM SOLVING APPROACH.

Contents

How to use the book

Mathematics is the language of precision. You use it whenever you deal with a question having to do with *how much* or *how many*. Mathematics is the language in which important ideas not only of science and business affairs, but also of recreation, sports, and personal activity are expressed. To use mathematics effectively you need not only develop your ability to solve problems and to compute accurately, but also to master mathematics as a language, understand its vocabulary, and be able to interpret and use the specialized methods of expressing mathematical ideas – graphs, formulas, and mathematical sentences. Here are some abilities you should achieve in each of these areas with the help of this book:

1. Learn to use systematic procedures in solving problems. To be skillful in problem solving you need a systematic procedure for attacking a problem. A series of steps to help you in solving problems is outlined and explained in the second chapter. To make the best use of them you should: (a) master each step so you can use it effectively; (b) always use each step in every problem you work; and (c) check up on yourself regularly to see if you are making proper use of each step. A problem scale in each chapter will give you a chance to see if you are neglecting any of the steps.

2. Develop accuracy in computations. Do you know how it feels to understand a problem perfectly and get the wrong answer because of an error in computation? Accuracy and speed in computation are not nearly so difficult to attain as many people think. Three things are important to

computational skill: (a) maintaining your skill in correct computational procedures; (b) writing your figures neatly and arranging your work in an orderly fashion; and (c) checking your answer.

If you wish to improve your mathematics, you must identify and correct your computational weaknesses. The *Inventory Tests* are designed to show where your weaknesses are in any computation. *Practice Exercises* are keyed to each Inventory Test to provide just the practice you need. They include *Examples* which show the correct form for use in practicing the computation.

3. Learn to understand and use the language of mathematics. To master a language you have to master its vocabulary and its special ways of expressing ideas. To help you do this, a list of *Words to Watch For* is placed at the beginning of each chapter. Each word is explained and used in the chapter. After taking a vocabulary test or finishing a chapter, go back and study the words again. The *Glossary* at the end of the book as well as the dictionary, will be useful for this purpose.

Graphs, formulas, and equations are also part of the language of mathematics. They are used extensively in the book in such a way as to help you learn to interpret them as well as to use them in expressing ideas. Remember always that understanding of the language of mathematics is essential where ideas are to be presented accurately, and mistakes may be costly. In your work you should always strive to understand the mathematics as well as to get the correct answer.

NUMBERS AND NUMERALS

WORDS TO WATCH FOR

base	*empty set*	*number*	*positional*
cardinal number	*exponent*	*number line*	*prime*
composite	*factor*	*numeral*	*set*
digit	*infinite*	*ordinal number*	*subset*
element	*natural number*	*place value*	*zero*

Mathematics is an important part of our language. Miss Adams' mathematics class learned this when Jim Stone reported on the vacation trip he and his family had enjoyed last summer. Starting from Chicago they had driven to Glacier and Yellowstone parks and had returned through the Black Hills. In telling about it Jim used numbers and numerals in a variety of ways.

1. They planned on 12 days of driving and figured that the entire trip would be about 3600 miles. What was the average number of miles they planned to drive per day? To find the average, divide the total number of miles by the number of days.

2. The distance from Chicago to Glacier Park by the northern route is 1561 miles. Jim prepared this table to show the distances between cities at which they planned to stop.

	From				
To	CHICAGO	LA CROSSE	FARGO	WILLISTON	HAVRE
La Crosse	268				
Fargo	653	385			
Williston	1075		422		
Havre	1384			309	
Glacier Park	1561				177

How far is it from Chicago to La Crosse, Wisconsin?

3. How far is it from La Crosse to Fargo, North Dakota?

4. On the second day of the trip they drove up the Mississippi from La Crosse on U.S. Highway 61 to St. Paul, and through the farming and lake country on U.S. 52. They decided not to stop in Fargo, but to go on for 77 miles through the flat Red River Valley wheat lands to Grand Forks. How far did they travel that day?

5. The third day's drive was 77 miles less than originally planned. How far was it to Williston from Grand Forks?

6. In Williston they had time to have the car serviced and to replace a tire that was showing wear. The lubrication cost 75¢ for grease and $1.50 for labor; the oil change required 6 quarts of No. 20 oil at 60¢ per quart. They purchased a No. 1 quality tire for $28, although a No. 2 quality tire would have been cheaper. How much was the total expense?

7. They arrived at Glacier Park entrance at noon of the fifth day. The 15-day permit cost $2, and they received sticker no. 378,492 to paste on the windshield. They were told that sticker no. 400,000 was sold 3 days later. How many cars had entered in the meantime?

8. On the second day in the park they drove over the spectacular Going-to-the-Sun Highway to Lake McDonald on the west side. They crossed the Continental Divide on this highway at Logan Pass at 6664 feet altitude. At the camp grounds on Lake McDonald the altitude is 4000 feet. This is how much lower than Logan Pass?

9. After four days in Glacier Park they drove to Yellowstone Park, where they again crossed the Continental Divide, this time at Craig Pass, at an altitude of 8262 feet. This is how much higher than Logan Pass?

10. Yellowstone Park is well known for its geysers, hot springs, and other evidences of volcanic activity. Most famous, of course, is Old Faithful geyser, which erupts at 65-minute intervals. On the veranda of the adjacent hotel is a bulletin board announcing the next scheduled eruption, which draws large crowds of tourists. If an eruption is scheduled for 8:00 A.M., how many times will the geyser erupt from 8:00 A.M. to 9:00 P.M. inclusive?

11. One of the beauty spots of Yellowstone is Yellowstone Lake, at an altitude of 7000 feet. This is how much higher than Lake McDonald?

12. In the Black Hills, they found that while there were mountains approaching 7300 feet altitude, the altitude of the highways is about 3000 feet. This is how much below the level of Craig Pass in Yellowstone?

13. In Exercises 1–12 are examples of numbers and numerals used in a variety of ways. Numbers used to tell how many objects are being considered are called *cardinal numbers.* List five examples of cardinal numbers in Exercises 1–12.

14. A number used to tell where in the counting process a given item comes (as first, second, etc.) is an *ordinal number.* Find four examples of ordinal numbers in the Exercises.

15. Numerals are parts of expressions, in the above exercises, in several different ways. A numeral stating number of "times" but not what *kind* of units expresses *frequency.* Find an example of this in the Exercises.

16. A numeral expressing *quantity* is used with a word or symbol such as miles, pounds, days, quarts, etc. Find five examples of this in the Exercises.

17. The words may also tell what is being measured or counted, as "yards of ribbon" etc. Thus you have both *quantity* and *kind.* Find an example of this in the Exercises.

18. Sometimes we have expressions stating *quantity, kind,* and *quality.* An example of this might be "2 dozen grade A eggs." Find two examples of this usage in the Exercises.

19. House, telephone, and license "numbers" are examples of the use of numerals to *identify* objects. Find four examples of this in the Exercises.

20. The use of numbers and numerals in the above illustrations is called *notation.* The use of numbers in adding, subtracting, multiplying, and dividing is called *computation.* Find an example of each kind of computation in the Exercises.

What is one-half of 8? Harold asked Martin this question, and Martin, of course, answered "4." "Wrong," said Harold, "It is 3." And writing 8 on the board, he divided it vertically in half like this: Ƹ 3. Thus he claimed he had shown that half of 8 is 3.

Harold's trick worked because he was talking about the *numeral* 8, which is the symbol he could write on the board. Martin was talking about the *number*, which he was using in the computation, $\frac{1}{2} \times 8$. A numeral is a name for the number. You cannot write a number; you write its name. Any number has many names. Besides 8, the number Martin was thinking of might be named in any of these ways:

$$7 + 1; \qquad 10 - 2; \qquad 2^3; \qquad 2 \times 4; \qquad \tfrac{16}{2}$$

VIII; eight; octo;

1. Write five other names for the number named by 6.

2. A system for writing numerals is called a *numeration system.* The system of numeration to which we are accustomed is called the *Hindu-Arabic.* It has several important characteristics that make it especially useful. One is that it makes use of ten basic symbols, called *digits*, which can be used to write any numeral, no matter how large a number it names. These symbols are: 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0. Selecting any three digits, can you guess how many three-digit numerals can be written, using each digit once? Which numeral will name the largest number?

3. A second characteristic is *place value.* The value of each digit in a numeral is determined by the position it occupies. The *value of a digit* is the number named by the digit multiplied by the value of its place in the numeral. In the numeral 5,162,703,914, the digits have values as follows, reading down in each column:

5	billion	0	ten thousand
1	hundred million	3	thousand
6	ten million	9	hundred
2	million	1	ten
7	hundred thousand	4	one or units

List, in the same way, the value of each digit in 3,245,905,302.

4. Another way to analyze the value of digits may be illustrated with the numeral 8043:

$$\begin{aligned} 8 \times 1000 &= 8000 \\ 0 \times 100 &= 0 \\ 4 \times 10 &= 40 \\ 3 \times 1 &= 3 \end{aligned}$$

Analyze each of the following in a similar manner:

a. 7309 **b.** 13,482 **c.** 130,256 **d.** 1,095,236

5. A third characteristic of the decimal system is its *additive* property. The sum of the values of the individual digits is what the numeral represents. For each of the numerals you analyzed in Exercise 4, a, b, c, and d, add the products and see if this is correct.

6. A fourth characteristic of the decimal system is that it is a *base-ten* system. This is what gives it the name *decimal*. The prefix "deci-" is derived from Latin and means "ten." Each place has associated with it a value 10 times that of the place immediately to its right. You saw this in analyzing the numerals in Exercise 4. Another way to state this property is that if a digit is "moved" one place to the left its value is multiplied by 10. Analyze each of the following, as in Exercise 4, and show that this is true:

a. 77 **b.** 444 **c.** 50,505 **d.** 30,003

7. For purposes of reading, a numeral in the decimal system is separated into "periods" of 3 digits each. Thus the numeral 3,245,915,382 is separated and named in the way which follows:

3,	245,	915,	382
billion	million	thousand	

This is read: 3 billion, 245 million, 915 thousand, 382. Write each of the following numerals, dividing each into periods by inserting commas. Then show, as above, how each is to be read.

a. 9343287647 **b.** 85723945 **c.** 384657396 **d.** 75789563492

8. 1000 billion is a trillion. 1000 trillion is a quadrillion. Continuing with multiples of 1000 are, in order: quintillion, sextillion, septillion, octillion, nonillion, and (if needed) decillion. The following numeral states the approximate weight of the earth in tons. Show how it is read: 6,594,126,820,000,000,000,000

9. You have encountered these prefixes before. Each of them refers to a number:

bi- in bicycle, biped, biennially, etc., means two
tri- in tricycle, tripod, triangle, etc., means three

What can you say about the meaning of?

quad-	quint-	sex-	sept-
oct-	non-	deci-	

10. Write the numeral that states the weight of the earth in pounds. Then show how it is read. (1 ton equals 2000 pounds.)

If we did not have our Hindu-Arabic system, could you find a way to tell your friends how old you are, or how many pupils are in your class? The ancient Greeks did this by using letters of their alphabet.

1	α'	alpha	10	ι'	iota	100	ρ'	rho
2	β'	beta	20	κ'	kappa	200	σ'	sigma
3	γ'	gamma	30	λ'	lambda	300	τ'	tau
4	δ'	delta	40	μ'	mu	400	υ'	upsilon
5	ϵ'	epsilon	50	ν'	nu	500	ϕ'	phi
6	ϛ′	digamma	60	ξ'	xi	600	χ'	chi
7	ζ'	zeta	70	o'	omicron	700	ψ'	psi
8	η'	eta	80	π'	pi	800	ω'	omega
9	θ'	theta	90	ϙ′	koppa	900	ϡ′	sampi

To show that the letters were to be read as numerals, the *accent sign* was used. For example, ϵ'. The ancient Greek system is called a *ciphered* numeration system, because there is a different cipher or symbol used for each number whose value is less than ten and for each of these numbers multiplied by ten or one hundred. While the position of a symbol in the ancient Greek system had nothing to do with its value, the symbols were usually written in *descending* order. That is, the symbol representing the greatest value is written first, and the symbol representing the least value is written last.

1. Thus 98 was written as ϙη'; 525 as $\phi\kappa\epsilon'$; 333 as $\tau\lambda\gamma'$; and 602 as $\chi\beta'$. How did the ancient Greeks write 22, 173, 79, 681, and 555?

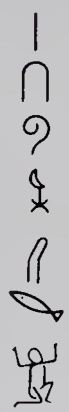
2. With a ciphered numeration system, it is possible to represent some numbers with fewer symbols than we use today. Write each of the following Hindu-Arabic numerals using ancient Greek symbols. Then underline those numerals that take fewer symbols when written with ancient Greek numerals.

880 50 300 35 473 405 40 111

3. Use the Hindu-Arabic system to write three, thirty, and three hundred. Does the position of the digit used help in determining its value, or is the value indicated by the symbol used, as in the ancient Greek system?

4. Add 4, 500, 90, and 800. What unique symbol in the Hindu-Arabic system makes it possible for us to use fewer different symbols? How does it make our computation much easier? Write an explanation of why it is more difficult to compute in the ancient Greek system than in the Hindu-Arabic system.

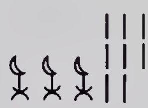
1. The ancient Egyptians used only a few symbols in their system of numeration. These represented 1 and powers of 10, as shown in the following chart.



Symbol	*Meaning*
vertical staff	1
heel bone	10
scroll	100
lotus flower	1000
pointing finger	10,000
burbot fish	100,000
man in astonishment	1,000,000

Their system was *additive.* That is, to write the numeral 30, they would repeat the symbol for 10. The three tens added together gave the value of 30 as follows:

Write these symbols in the Hindu-Arabic system of numeration:

, , , and 

2. Write the Egyptian symbols for each of the following numerals:

a. 3 **b.** 9 **c.** 341 **d.** 879 **e.** 3198

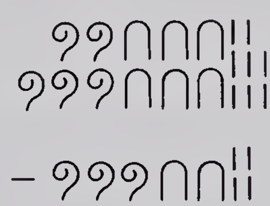
3. The position of the symbols was not important in the Egyptian system. Thus the numeral 22 could be written in several ways:

 or 

Express 22 in two other ways, using Egyptian symbols.

4. Addition and subtraction with Egyptian symbols is not difficult.

Add: 

Subtract: 

Write your answers using Egyptian symbols. Then express your answers in Hindu-Arabic symbols.

Still another system of numeration that had the additive property was the Ancient Roman system. Except for the symbols for 1 and 5, each symbol represents a value that is a *multiple* of 10.

Remember! A multiple of 10 refers to numbers that are *exactly divisible* by 10. That is, there is no remainder. In the following listing, 50, 100, 500, and 1000 are multiples of 10.

I	V	X	L	C	D	M
1	5	10	50	100	500	1000
		(10×1)	(10×5)	(10×10)	(10×50)	(10×100)

By putting a bar over a symbol, the number it named became 1000 times as great. Therefore, each of the following is a multiple of 1000.

$\overline{\mathrm{V}}$	$\overline{\mathrm{X}}$	$\overline{\mathrm{L}}$
5000	10,000	50,000
(5×1000)	(10×1000)	(50×1000)
$\overline{\mathrm{C}}$	$\overline{\mathrm{D}}$	$\overline{\mathrm{M}}$
100,000	500,000	1,000,000
(100×1000)	(500×1000)	(1000×1000)

1. Write these numerals in the Ancient Roman system:

a. 4 **b.** 8 **c.** 84 **d.** 494 **e.** 50,180 **f.** 11,956

2. Write each of the following in Hindu-Arabic numerals.

a. VIII
b. XXXXVIIII
c. DCCCCLXXXXIIII
d. CCCCLXXXXVIIII
e. $\overline{\mathrm{M}}$VI
f. $\overline{\mathrm{V}}$II

3. Without the use of Hindu-Arabic numerals:

a. Add CCXXXIII and CCCXXXXVI
b. Subtract CCCXXIIII from DLXVIII

4. Compare your work in Exercise 3 above with Exercise 4 on page 7. Your answers should be the same. Do you find either system of numeration (Egyptian or Ancient Roman) more convenient for addition and subtraction? If so, explain why you believe that system is more convenient.

5. Multiplication in the Egyptian or Ancient Roman system is quite difficult. See if you can perform these multiplications without changing the symbols into Hindu-Arabic numerals.

a. 𓍢𓎆𓎆𓏺𓏺𓏺 by 𓆼𓎆𓏺𓏺𓏺 **b.** LXVIII by DLXI

The Roman numerals with which you are familiar and that are used today on clocks and in inscriptions on buildings were a later development of the Ancient Roman system. This later development included a principle of subtraction. Following are the instances of this principle.

(a)

IV	IX
$5 - 1 = 4$	$10 - 1 = 9$
XL	XC
$50 - 10 = 40$	$100 - 10 = 90$
CD	CM
$500 - 100 = 400$	$1000 - 100 = 900$

In the Ancient Roman system these would be written as follows:

(b)

IIII	VIIII	XXXX	LXXXX	CCCC	DCCCC
4	9	40	90	400	900

The principle of subtraction certainly simplifies the writing of numerals because fewer symbols are needed. Also, the position of the symbols becomes more significant. For example, if the symbols in (a) are rearranged as follows, the principle of addition is used:

(c)

VI	XI
$5 + 1 = 6$	$10 + 1 = 11$
LX	CX
$50 + 10 = 60$	$100 + 10 = 110$
DC	MC
$500 + 100 = 600$	$1000 + 100 = 1100$

There are but *six* cases of the principle of subtraction, but there are many instances of the principle of addition.

How would you write Roman numerals for 944 and for 964?

(d)

$$\underbrace{\text{C}\quad\text{M}}_{1000 - 100}\quad \underbrace{\text{X}\quad\text{L}}_{50 - 10}\quad \underbrace{\text{I}\quad\text{V}}_{5 - 1}$$

$$\underbrace{900 \quad + \quad 40 \quad + \quad 4}_{944}$$

(e)

$$\underbrace{\text{C}\quad\text{M}}_{1000 - 100}\quad \underbrace{\text{L}\quad\text{X}}_{50 + 10}\quad \underbrace{\text{I}\quad\text{V}}_{5 - 1}$$

$$\underbrace{900 \quad + \quad 60 \quad + \quad 4}_{964}$$

1. Write the names for each of the following symbols in the Hindu-Arabic system:

   a. XVI
   b. XLII
   c. CMIV
   d. MCMIX
   e. XLV
   f. $\overline{\mathrm{X}}$X
   g. MCMLXVII
   h. MMMXL
   i. $\overline{\mathrm{C}}$V
   j. DLVII
   k. CMXCIX
   l. MDCXVI

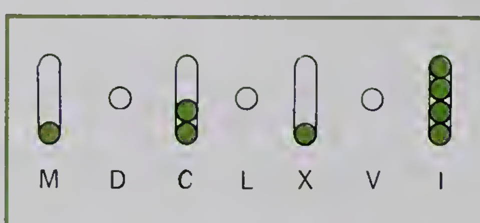
2. The following are expressed in the Ancient Roman system of numeration. Write the names for them in later Roman numerals.

   a. MMCCCCXXXXIIII
   b. MCCCCXXXXIII
   c. MMMMVIIII

3. Write the present year using the later Roman system. Write it using the Ancient Roman system.

4. If you are now XVI years old, how old will you be in the year MCMXCIX? Write the answer using later Roman symbols.

5. In comparison with our Hindu-Arabic system, the earlier systems of Greece, Egypt, and Rome were not convenient for computation. Calculations required special devices. One such device was a table covered with sand. Grooves were made in the sand into which pebbles could be placed. Each groove represented a certain unit, with the number of pebbles in the grooves indicating how many times the unit was to be taken. However, when five pebbles appeared in the groove for C, X, or I, they were all removed and only one pebble was placed in the groove to the left. Why? Why is there room for only one pebble in each of the grooves labeled V, L, and D in the Figure below?



6. Draw a sketch to show how the above Figure will look if you increase the value of the number it represents by 1.

7. Draw a sketch to show how the table will look if you add MMMCXX to the number represented in the Figure above.

8. Draw a sketch to show how the table will look if you subtract MMMDCCCXXV from the answer to Exercise 7.

9. Now make a sketch to show the table after you add DCCV to the answer to Exercise 8. If you made no mistakes you should have the number you started with in the Figure.

You will notice Roman numerals used frequently today in special places. The production date of a motion picture is expressed in Roman numerals. Look for it right after the cast of characters next time you attend a movie. Large buildings, churches, and monuments often include a cornerstone with the year chiseled in Roman numerals. You may have also seen Roman numerals on clock faces. Roman numerals today are used more for design than for mathematics.

1. Find five buildings in your community which have cornerstones showing the year the building was constructed. How many had the date expressed in Roman numerals?

2. Robert found the Roman numeral MCMXXXVI on the cornerstone of the library. Write the date as a Hindu-Arabic numeral.

3. The seven basic symbols used in the Roman system are I, V, X, L, C, D, M. Write the value of each in our decimal system.

4. Write the Hindu-Arabic numerals for:

a. MC	**e.** CM	**i.** CDXLIV
b. XLI	**f.** MCD	**j.** $\overline{\text{C}}$XXXIX
c. XXXIX	**g.** MDC	**k.** CMIII
d. CI	**h.** MCMLV	**l.** DCCCVIII

5. Write the Roman numerals for:

a. 28	**e.** 10,449	**i.** 1866
b. 86	**f.** 1961	**j.** 50,001
c. 666	**g.** 1975	**k.** 2430
d. 505	**h.** 1492	**l.** 5609

6. Write the Roman numerals for:

a. seventy-eight
b. five hundred forty
c. nine hundred ninety-nine
d. one thousand nine hundred sixty-five
e. two thousand three hundred forty-four
f. fifty thousand ninety-six
g. six hundred forty-nine
h. four hundred twenty-six
i. one thousand eight hundred sixty

7. Using I, V, and X only once, what is the largest possible number that you can name? Using these same symbols, what is the smallest possible number that you can name?

A simple way of interpreting the base-ten property is in terms of *powers*, or *exponents*. The exponent tells us how many times the base (in this case, 10) is to be used as a *factor*. Remember! The word "factors" refers to the numbers used when we multiply. When we state that $3 \times 2 = 6$, the 3 and 2 are factors of 6, and 6 is the *product*. (Note that 1 and 6 are also factors of 6.) In other words, since 6, 3, 2, and 1 divide 6 exactly (no remainder), then they are factors of 6.

EXAMPLE

Notice that the exponent is written above and to the right of 10.

$$\begin{aligned} 10^2 &= 10 \times 10 &&= 100 \\ 10^3 &= 10 \times 10 \times 10 &&= 1{,}000 \\ 10^4 &= 10 \times 10 \times 10 \times 10 &&= 10{,}000 \end{aligned}$$

1. Do you see a pattern in the above Example? State this pattern in your own words. Use this pattern to find the value of:

a. 10^5 **b.** 10^6 **c.** 10^7 **d.** 10^9

2. As the exponent of 10 *increases* by 1, the previous number is multiplied by 10. Therefore, going in the opposite direction, as the exponent of 10 *decreases* by 1, the previous number is divided by 10. Therefore, if $10^4 = 10{,}000$, then $10^3 = 10{,}000 \div 10 = 1{,}000$. Then, $10^2 = ?$

3. If the system is to work according to the rule stated in Exercise 2, what should be the value of 10^1? of 7×10^1? of 3×10^1?

EXAMPLE

Using exponents to indicate powers of 10, let us analyze 69,730.

$$\begin{aligned} 6 \times 10^4 &= 6 \times 10{,}000 = 60{,}000 \\ 9 \times 10^3 &= 9 \times 1{,}000 = 9{,}000 \\ 7 \times 10^2 &= 7 \times 100 = 700 \\ 3 \times 10^1 &= 3 \times 10 = \underline{\quad 30} \\ & \qquad\qquad\qquad 69{,}730 \end{aligned}$$

4. To go one step further, if the rule "works" then how about 10^0? Using the rule of Exercise 2, we should divide 10^1 by 10 to find 10^0. Therefore, $10^0 = ?$ We will define any natural number, 1, 2, 3, 4, etc., raised to the zero power, 1^0, 2^0, 3^0, 4^0, etc., to be equal to 1.

5. What is the value of 7^0? of 4×8^0? of 6×12^0?

Probably our numeration system is a decimal system because people first learned to count on their ten fingers. When they used the sand table it was natural to continue with base ten. To represent ten in any one groove we have to remove the pebbles in that groove and place one pebble in the next groove to the left. Each groove held only 9 pebbles.

1. Suppose you have a sand table with grooves only long enough to hold four pebbles. If there are four pebbles in the first groove as in Figure 1 below, draw a picture of how the table will look if you add 1 to the number represented on the table.

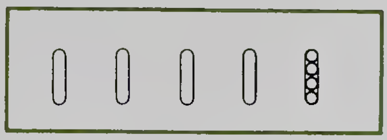


Figure 1

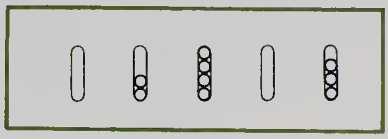


Figure 2

2. Draw a picture of how the table would look if you were to add three to the number represented on the table in Figure 2 above.

3. Since the grooves are only long enough to hold four pebbles instead of nine, any time we have five pebbles to put in a given groove we must remove them and put one pebble in the groove to the left of it. What is the base of the number system that this sand table represents? Remember! When the grooves held nine pebbles, the base was ten.

4. Draw a picture of a sand table with 6 grooves where each groove holds only four pebbles, and label the grooves using exponents.

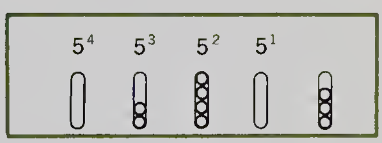
5. Using the picture you drew in Exercise 4, write the numeral in base five for each of these:

a. five^4 **b.** five^3 **c.** five^2 **d.** five

Note: five^5 becomes 100000 in base five. However, five$^5 = 5 \times 5 \times 5 \times 5 \times 5 = 3125$ in base ten. Therefore, 100000 in base five $= 3125$ in base ten or $100000_{\text{five}} = 3125_{\text{ten}}$.

6. Using positional notation, the number represented on the table below would be written as 2403_{five}. This means

$$\begin{aligned}
2 \times 5^3 &= 2 \times 5 \times 5 \times 5 &= 250\\
4 \times 5^2 &= 4 \times 5 \times 5 &= 100\\
0 \times 5^1 &= 0 \times 5 &= 0\\
3 \times 5^0 &= 3 \times 1 &= \underline{3}
\end{aligned}$$



353 base ten or 353_{ten}

Therefore $2403_{\text{five}} = 353_{\text{ten}}$

Explain why we have to use a zero in the numeral 2403.

7. The sand table in Exercise 6 is a base-five table while the ones in Figure 3 below are base-ten tables. Explain how you can tell this from the Figures.

8. Now suppose you had the number represented in Figure 3a below on a base-ten table. Draw a sketch to show how the same number would be represented on a base-five table.

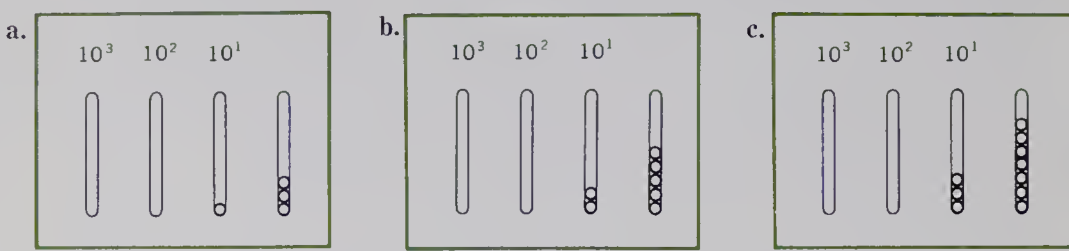
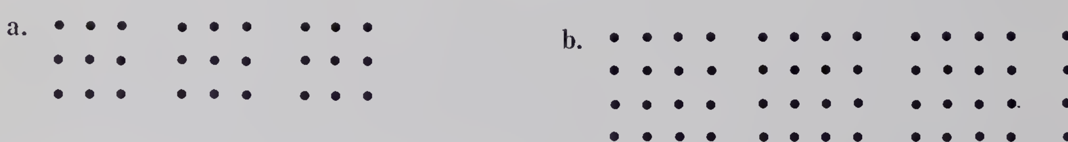


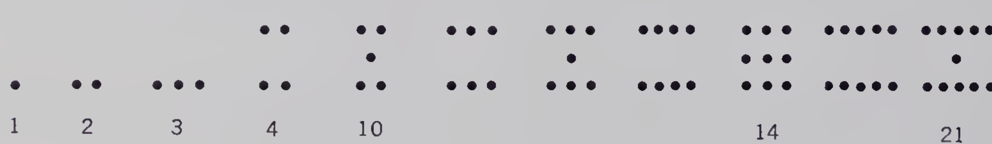
Figure 3

9. Draw sketches to show how the numbers represented in Figures 3b and 3c would be represented on a base-five table. After you have drawn the sketches, look at your sketches and write base-five numerals to represent the numbers.
HINT: See Exercise 6 on page 13.

10. Draw sketches to represent the number of dots in a and b below on a base-ten sand table, and also on a base-five sand table. Then express the number in base ten and also in base five.



11. Copy the eleven groups of dots below, and fill in the correct numerals where they are missing to show how each number would be expressed in base five.



12. In Exercise 11 you have written the numerals one through eleven in base five. From this we see that:

5 (base ten) = 10 (base five) or $5_{\text{ten}} = 10_{\text{five}}$, and that
11 (base ten) = 21 (base five) or $11_{\text{ten}} = 21_{\text{five}}$
Then 12 (base ten) = ? (base five)
13 (base ten) = ? (base five)
14 (base ten) = ? (base five)

A. *Add:*

1.	3.	5.	7.
7218	732,517	\$ 57.75	\$200.50
4026	50,831	7.95	35.09
183	6,130	415.56	115.09
3760	498	87.50	700.00
595	26	1500.00	56.63

2.	4.	6.	8.
534	6	\$ 98.09	\$ 75.00
389	38	2.07	3.05
1298	165	108.45	245.05
703	1,945	59.05	90.80
472	28,347	413.87	803.03

B. *Subtract:*

1.	3.	5.	7.
135	415	625	113
− 90	− 85	− 156	− 34

2.	4.	6.	8.
509	800	790	403
− 352	− 56	− 631	− 399

C. *Multiply:*

1.	3.	5.	7.
305	702	450	320
× 18	× 67	× 203	× 155

2.	4.	6.	8.
812	260	575	400
× 230	× 190	× 600	× 400

D. *Divide:*

1. 686 ÷ 14	**3.** 3230 ÷ 95	**5.** 1045 ÷ 19
2. 6102 ÷ 27	**4.** 1131 ÷ 13	**6.** 2584 ÷ 68

If you need more practice, use the Practice Exercises on the following pages. If not, you may work in the Experts' Corner on page 20.

A. *Addition*

EXAMPLE

48	What is the sum of the numbers named in the units column?
35	How many tens are "carried" to the tens column?
41	How many hundreds have been "carried" to the hundreds
53	column?
177	

Add:

1.	**2.**	**3.**	**4.**	**5.**
74	30	72	84	47
65	53	91	39	85
57	19	46	24	92
89	40	94	86	38
57	38	95	46	33

6.	**7.**	**8.**	**9.**	**10.**
59	92	50	63	94
72	59	28	80	58
61	87	58	57	80
66	18	52	81	92
92	23	36	24	67
35	19	57	45	30
82	62	80	76	48

11.	**12.**	**13.**	**14.**
38	87	356	944
15	38	577	944
71	37	259	537
61	71	757	189
21	53	750	209
57	45	152	815
63	43	495	773

15.	**16.**	**17.**	**18.**
216	\$53.78	\$ 102.09	\$.56
694	75.56	52.28	.32
356	68.37	53.73	.29
187	.40	89.75	.77
382	50.73	.24	.51
948	.16	36.24	.74
251	26.56	1190.57	.85

B. *Subtraction*

EXAMPLE

$$\begin{array}{r} 623 \\ -\ 349 \\ \hline 4 \end{array}$$

(1) Since we cannot subtract 9 from 3, 23 is rewritten as 10 + 13. Then ? − 9 = ?

$$\begin{array}{r} 623 \\ -\ 349 \\ \hline 74 \end{array}$$

(2) Since we cannot subtract 40 from 10, 610 is rewritten as 500 + 110. Then ? − 40 = ?

$$\begin{array}{r} 623 \\ -\ 349 \\ \hline ?74 \end{array}$$

(3) Finally, ? − 300 = ?

Note: Steps (1) and (2) in the past were referred to as "borrowing." As the example illustrates, borrowing really means writing another name for the number. The numeral 623 was rewritten as 500 + 110 + 13 so that we could subtract 300 + 40 + 9.

Subtract:

1. 254 − 116	**8.** 1756 − 1578	**15.** \$19.07 − 17.45	**22.** \$16.80 − 7.05
2. 518 − 319	**9.** 1323 − 1005	**16.** \$15.50 − 9.55	**23.** \$64.20 − 47.77
3. 349 − 155	**10.** 6020 − 4131	**17.** \$92.07 − 43.25	**24.** \$41.00 − 40.01
4. 843 − 607	**11.** 1911 − 855	**18.** \$38.50 − 19.49	**25.** \$305.05 − 260.06
5. 407 − 311	**12.** 9030 − 7500	**19.** \$42.00 − 19.07	**26.** \$340.70 − 115.19
6. 613 − 425	**13.** \$17.05 − 15.50	**20.** \$50.32 − 43.33	**27.** \$570.90 − 480.05
7. 3214 − 1621	**14.** \$35.00 − 18.05	**21.** \$25.03 − 9.11	**28.** \$600.00 − 115.23

C. *Multiplication*

EXAMPLES

1. Complete the explanations:
   First multiply 435 × $\underline{\ ?\ }$
   Then multiply 435 × $\underline{\ ?\ }$
   The units digit of the second partial product is written under the $\underline{\ ?\ }$ digit of the first partial product. Why?

$$\begin{array}{r} 435 \\ \underline{\times\ 67} \\ 3045 \\ \underline{2610} \\ 29145 \end{array}$$

2. First put down a 0 for the $\underline{\ ?\ }$ in the multiplier. Why?
   Then find the product of 312 × $\underline{\ ?\ }$
   Write this product in front of the $\underline{\ ?\ }$ you had just put down.

$$\begin{array}{r} 312 \\ \underline{\times\ 60} \\ 18720 \end{array}$$

3. First find the product of 417 × $\underline{\ ?\ }$
   Then put down $\underline{\ ?\ }$ for the $\underline{\ ?\ }$ in the multiplier.
   Then find the product of 417 × $\underline{\ ?\ }$
   Write the product in front of the $\underline{\ ?\ }$ you just wrote.
   Complete the exercise.

$$\begin{array}{r} 417 \\ \underline{\times\ 506} \end{array}$$

Multiply:

1. $\begin{array}{r} 221 \\ \underline{19} \end{array}$
2. $\begin{array}{r} 532 \\ \underline{56} \end{array}$
3. $\begin{array}{r} 818 \\ \underline{35} \end{array}$
4. $\begin{array}{r} 751 \\ \underline{44} \end{array}$
5. $\begin{array}{r} 542 \\ \underline{47} \end{array}$
6. $\begin{array}{r} 815 \\ \underline{75} \end{array}$
7. $\begin{array}{r} 814 \\ \underline{70} \end{array}$
8. $\begin{array}{r} 453 \\ \underline{60} \end{array}$
9. $\begin{array}{r} 371 \\ \underline{80} \end{array}$
10. $\begin{array}{r} 109 \\ \underline{96} \end{array}$
11. $\begin{array}{r} 601 \\ \underline{48} \end{array}$
12. $\begin{array}{r} 768 \\ \underline{50} \end{array}$
13. $\begin{array}{r} 431 \\ \underline{602} \end{array}$
14. $\begin{array}{r} 516 \\ \underline{705} \end{array}$
15. $\begin{array}{r} 834 \\ \underline{306} \end{array}$
16. $\begin{array}{r} 360 \\ \underline{802} \end{array}$
17. $\begin{array}{r} 419 \\ \underline{503} \end{array}$
18. $\begin{array}{r} 238 \\ \underline{702} \end{array}$
19. $\begin{array}{r} 807 \\ \underline{450} \end{array}$
20. $\begin{array}{r} 450 \\ \underline{710} \end{array}$
21. $\begin{array}{r} 2250 \\ \underline{813} \end{array}$
22. $\begin{array}{r} 3050 \\ \underline{707} \end{array}$
23. $\begin{array}{r} 1790 \\ \underline{506} \end{array}$
24. $\begin{array}{r} 6250 \\ \underline{415} \end{array}$
25. $\begin{array}{r} \$19.77 \\ \underline{402} \end{array}$
26. $\begin{array}{r} \$42.29 \\ \underline{803} \end{array}$
27. $\begin{array}{r} \$31.05 \\ \underline{205} \end{array}$
28. $\begin{array}{r} \$40.20 \\ \underline{606} \end{array}$

D. *Division*

Look at the second digit in the divisor. If it is 4 or less, use the number named by the first digit as the trial divisor. If it is 5 or more, increase the value of the first digit by 1, and use it as a trial divisor.

Look at Example 1. Where does 18 come from? Where does 6 come from? Complete this part of the division.

Look at Example 2. Where does 24 come from? Where does 4 come from? Complete this part of the division. Does the trial quotient need correction?

In each of the following, find only the first digit in the quotient. Correct the trial quotient if necessary.

EXAMPLES

1.
$$\begin{array}{r} 3 \\ 573\overline{)18924} \end{array}$$
$18 \div 6 = 3$

2.
$$\begin{array}{r} 6 \\ 447\overline{)24591} \end{array}$$
$24 \div 4 = 6$

1. $685 \div 19$
2. $1821 \div 57$
3. $3682 \div 69$
4. $4227 \div 73$
5. $6482 \div 34$
6. $1764 \div 56$
7. $4622 \div 66$
8. $1921 \div 55$
9. $3462 \div 75$
10. $2959 \div 46$
11. $1831 \div 65$
12. $4932 \div 71$

EXAMPLE

a. Use 6 as a trial divisor. Why?

b. There are 3 6's in 20. (Actually, we are saying that there are 300 60's in 20,000.)
Write 3 directly above 2. Why?

c. $202 - 168 = 34$
Where is 34 written? Where does 1 come from?

d. Use 6 as a trial divisor again. Complete the explanation.

$$\begin{array}{r} 361 \\ 56\overline{)20216} \\ \underline{168} \\ 341 \\ \underline{336} \\ 56 \\ \underline{56} \end{array}$$

Divide:

1. $1372 \div 28$
2. $5796 \div 63$
3. $4018 \div 49$
4. $2233 \div 77$
5. $3599 \div 59$
6. $3051 \div 27$
7. $3393 \div 39$
8. $5976 \div 83$
9. $1615 \div 95$
10. $2090 \div 38$
11. $5394 \div 62$
12. $3055 \div 47$
13. $4602 \div 78$
14. $14{,}857 \div 179$

1. List the digits, including zero, that make up a base-five numeration system.
2. Using base five, write your own age, the number of pupils in your class, and the number of windows in your classroom.
3. Copy the base-five addition table below. Using base five, check the numerals already filled in to see if they are correct, and fill in the blank spaces with the correct numerals.

Addition Table

+	0	1	2	3	4
0		1	2	3	4
1	1	2		4	10
2	2	3	4		11
3	3		10	11	12
4	4	10		12	

Multiplication Table

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	
2	0		4	11	
3	0	3		14	22
4		4	13		31

BASE FIVE

4. Copy the base-five multiplication table above. Using base five, check to see if the numerals already filled in are correct, then fill in the blank spaces with the correct numerals.
5. 10 base five = 5^1 base ten; 100 base five = 5^2 or 25 base ten; 1000 base five = _?_ base ten; and 10000 base five = _?_ base ten.
6. An easy way to change a numeral expressed in base ten to some other base is simply by *repetitive* division. For example, to change 117 base ten to base five we proceed as follows:

```
5 | 117
 5 | 23   Remainder 2
  5 | 4   Remainder 3
      0   Remainder 4
```

The desired numeral, 432 base five, is expressed by the remainders.

117 base ten = 432 base five. Prove this is right and show your work.
HINT: Does $1 \times 10^2 + 1 \times 10^1 + 7 \times 10^0 = 4 \times 5^2 + 3 \times 5^1 + 2$?

7. To change 19 base ten to base five:

```
5 | 19
 5 | 3   Remainder 4
     0   Remainder 3
```

19 base ten = 34 base five

Change the following base-ten numerals to base five: 247, 16, 9, and 4. Show your work and check your answers as above.

In finding a trial divisor, it is useful to know, without actually dividing, whether one number can be divided by another without a remainder.

You know that even numbers can be divided exactly by 2, while odd numbers cannot. There are several other simple rules. For example, a number can be divided exactly by 3 if the sum of the numbers named by its digits is a multiple of 3.

EXAMPLE

Is 429 divisible by 3?

$4 + 2 + 9 = 15$

Since 15 is a multiple of 3, then 429 is divisible by 3.

Proof: $429 \div 3 = 143$

1. Test each of the following to see if it is divisible by 3. If it is, find the quotient to see if you are right.

a. 419	**e.** 856	**i.** 540	**m.** 612
b. 276	**f.** 975	**j.** 617	**n.** 256
c. 432	**g.** 123	**k.** 832	**o.** 145
d. 529	**h.** 246	**l.** 912	**p.** 541

2. If the sum of the numbers named by its digits is a multiple of 9, the number is divisible by 9. Which of the numbers named in Exercise 1 are divisible by 9?

3. If an even number is divisible by 3, the number is divisible by 6. Why? Which of the numbers named in Exercise 1 are divisible by 6?

4. If an even number is divisible by 9, the number is divisible by 18. Why? Which of the numbers named in Exercise 1 are divisible by 18?

5. If its last two digits name a number divisible by 4, the number is divisible by 4. Which of the numbers named in Exercise 1 are divisible by 4?

6. If the units digit is a 5 or a 0, the number is divisible by 5. Which of the numbers named in Exercise 1 are divisible by 5?

7. How could you tell, without dividing, if a number is divisible by 15? Which of the numbers named in Exercise 1 are divisible by 15?

8. A number that can be divided into another without a remainder is a *factor* of that number, as we have already seen. Copy this table, and indicate by + if each of the numbers 2, 3, 4, 5, and 9 is a factor of the numbers listed in a–f. If it is not, write "no."

Factors to test

	2	3	4	5	9
a. 175					
b. 321					
c. 258					

Factors to test

	2	3	4	5	9
d. 536					
e. 912					
f. 411					

We have many names that we use for a collection of objects. We speak of a group of people, a herd of cattle, a flock of sheep, or a set of dishes. The word *set* is a useful term and is commonly used to name a collection. We speak of a set of golf clubs, a set of encyclopedias, a set of silverware, etc. When we refer to a set in mathematics, we are talking about a collection of objects that is defined so that we can readily tell which things belong to the set and which do not. The following problems will help to illustrate what we mean by a set.

1. Tom always carried with him his billfold, jackknife, comb, handkerchief, and his lucky nickel. These objects can be referred to as a set. How many members are there in the set? This is called the *cardinal number* of the set. (You met cardinal numbers earlier on page 3.)

2. Remember, we should be able to determine easily whether or not a given object belongs to a set. If we cannot tell, then in fact we do not have a set. Does Tom's notebook belong to the set of Exercise 1?

3. The objects that belong to a set are called the *elements* or the *members* of the set. In Exercise 1, the set was described by naming each individual member. Sometimes it is more useful *not* to list each member but instead, describe the set in words as follows:

{the natural numbers from 1 to 9, including both 1 and 9}

Notice that we use *braces* to enclose the description of the set. To simplify our discussion of this set, let us call it set A. We will use capital letters to name sets. What is the cardinal number of set A?

4. Which of the following name elements of set A?

a. 6
b. 15
c. 0
d. 3^2
e. $5+2$
f. $12-9$
g. 3×8
h. 5^0
i. $10^2 \div 10^2$
j. 8×3

5. Instead of describing the elements in a set as we did in set A, we can simply list them. For example:

$$\text{set } B = \{1,3,5,7,9,11,13,15,17,19\}$$

How many elements are in set B, or in other words, what is the "cardinal number" of set B?

6. The elements in a set may be listed in any order. For example, set $C = \{11,13,17\}$ can also be written as set $C = \{13,17,11\}$. Name set C using still a different order of the elements.

7. Notice that we use braces whether we list the elements in a set (set B) or use the description of the set (set A). Describe set B in words.
8. Which of the following name elements of set B?

a. 17	**f.** $16 - 3$
b. 1	**g.** 21
c. 10^0	**h.** 19
d. 6	**i.** $119 - 100$
e. $36 \div 2$	**j.** 0

9. Compare the elements in the following set with those of set B in Exercise 5.

$$\text{set } D = \{1,3,5,7,9\}$$

Because *each* of the elements of set D is also an element of set B, we call set D a *subset* of set B. Which of the following are subsets of set D?

a. $\{3,5,7\}$	**d.** $\{9,7,5,3,1\}$
b. $\{13,11,9\}$	**e.** $\{15,13\}$
c. $\{5\}$	**f.** $\{17,1,7,5\}$

10. You probably noticed that all the sets in Exercise 9 were subsets of set B. Write 3 other subsets of set B.

In the following exercises consider the set

$$F = \{2,4,6,8,10,12,14,16,18,20\}$$

List the elements of each of the following: (Do not forget to use braces!)

11. The subset of set F whose elements are factors of 6
HINT: See page 21.
12. The subset of set F whose elements are multiples of 5
HINT: See page 8.
13. The subset of set F whose elements are greater than 7 and less than 11
14. The subset of set F whose elements are greater than 8 and less than 16 and that are also multiples of 4
15. Consider the following sets:

$$A = \{2,4,6,8,10,12\} \qquad B = \{3,6,9,12\}$$

Combine the elements of sets A and B into one set. If an element occurs in both A and B, you only need to list it once.

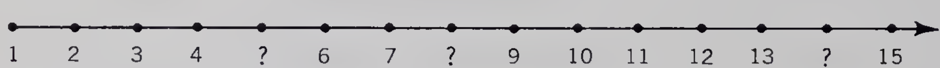
In mathematics we use the word "set" most often to refer to a set of numbers, even though we can have sets of points, lines, letters, or miscellaneous objects.

1. The set of numbers we use for counting, 1, 2, 3, 4, 5, etc., is called the set of *natural numbers.* We will use **N** to stand for the set of natural numbers. Is 17 a natural number?

2. Can you name the natural number that has the greatest value? Regardless of what number you name, a greater number can be obtained simply by adding 1 to the number you name. Therefore, since there is no greatest natural number, we say that the set of natural numbers is *infinite.* Name a natural number which is:

a. greater than 1,000
b. greater than 1,000,000
c. greater than 1,000,000,000

3. Although we cannot list all the natural numbers, we can represent **N** by using a *number line.* The following is an illustration of a number line. Notice that each natural number is associated with a "dot" (actually a point) on the number line. Each dot that represents a natural number is spaced equally. That is, the space from 2 to 3 is the same as the space from 9 to 10. Is the distance from 4 to 5 equal to the distance from 3 to 4?



4. Since the number line above represents **N**, the first dot, or starting point, is 1. Since **N** is infinite, we use an arrowhead on the right end of the number line to show this. What natural numbers do the question marks in the illustration represent?

5. We are often concerned with subsets of **N**. For example, the set of the *even* natural numbers that we will call set E,

$$E = \{2,4,6,8,10,12,14,16, \ldots\}$$

and the set of the odd natural numbers that we will call set O,

$$O = \{1,3,5,7,9,11,13,15, \ldots\}$$

are discussed frequently. Notice the three dots, . . . , after the 16 in set E and after the 15 in set O. These three dots have the same meaning as the arrowhead on the number line above. What do the three dots mean? If we can actually count the number of elements in a set, then the set would not be infinite. It would be *finite.*

The problems on this page will ask you to form subsets of the natural numbers according to certain conditions.

List the elements of each of the following subsets of **N**. Do not forget the braces or the three dots if the set is infinite!

1. The set of natural numbers
2. The set of natural numbers that are factors of 8
3. The set of natural numbers greater than 17 and less than 21
4. The set of natural numbers that are factors of 21
5. The set of even natural numbers that are factors of 20
6. The set of even natural numbers greater than 4 and less than 12
7. The set of odd natural numbers that are factors of 36
8. The set of even natural numbers that are factors of 28
9. The set of natural numbers less than 50 that are multiples of 13
Note: Include 13 in this set because it is exactly divisible by itself.
10. The set of natural numbers greater than 10 and less than 60 that are multiples of 12
11. The set of natural numbers that are factors of *both* 7 and 11
12. The set of natural numbers greater than 1 that are factors of 7
13. The set of natural numbers greater than 1 and less than 7 that are factors of 7

In the following exercises write in words a descriptive name for each of the listed sets

14. {9,10,11,12}
15. {3,6,9,12,15}
16. {11,22,33,44,55,66}
17. {1,2,3,6}
18. {1,4,9,16,25}
19. {1,8,27,64}

From set $A = \{1,2,3,4,5,6,7,8,9\}$, select the element or elements that fit each of the following:

20. The numbers greater than 6
21. The numbers less than 1
22. The numbers greater than 2 and less than 7
23. The numbers which are factors of 12
24. The factors of 36 which are greater than 3

If we add zero to our set of natural numbers, we have the set of *whole numbers,* which we will call the set **W**.

$$\mathbf{W} = \{0,1,2,3,4,5,6, \ldots\}$$

We can use a number line to show the set of whole numbers.

Note that zero is the starting point on this number line.

1. Considering the set of whole numbers, **W**, and the set of natural numbers, **N**, which is the subset of the other?

2. Is the set of whole numbers infinite?

3. In which direction is the number line extended to show that the set is infinite?

List the elements of each of the following subsets of **W**.

4. The set of odd numbers

5. The set of whole numbers less than 7

6. The set of whole numbers greater than 10

7. The set of whole numbers less than 1

8. The set of whole numbers that are multiples of 5

Remember! A number is a multiple of another number if it is exactly divisible by that number. Thus 0 is a multiple of 5, since $0 \div 5 = 0$.

9. The set of whole numbers that are multiples of 6

10. The set of whole numbers that are not natural numbers

11. As we mentioned earlier, any set A, all of whose members are also members of another set B, is a subset of B. (See page 23.) The symbol for "is a subset of" is $\subset$.

For example, let set $A = \{1,3,5,7,9\}$ and set $B = \{3,5,7\}$.

Then: $$B \subset A$$

Write in words a description of five subsets of **N**, and use the symbol $\subset$ to show that each is a subset of **N**.

12. If set $E = \{$all even natural numbers$\}$, write in words a description of five subsets of E, and use the symbol $\subset$ to show that each is a subset of E.

13. If set $O = \{$all odd numbers$\}$, write in words a description of five subsets of O, and state the relationship using the symbol $\subset$.

In our set of natural numbers, we have numerous subsets, as we have already seen. We have spoken of the set of even numbers, the set of odd numbers, and the set of numbers divisible by 5, to name a few.

1. List the set of natural numbers that are factors of 24.
2. List the set of natural numbers that are factors of 21.
3. List the set of natural numbers that are factors of 19.
4. In **N**, there are members each of which has only two factors, 1 and itself. For example, the only factors of 7 are 1 and 7. Natural numbers that have this property are called *prime numbers* or simply *primes*. List the set of natural numbers less than 10 that are prime. (Did you include 1 in your answer?)
5. We know that 1 is not a prime number since it has only one factor in the set of natural numbers. Is 2 a prime number?
6. It is sometimes difficult to determine if a given number is prime. The obvious test is simply to find factors of the number in question. Therefore, when we ask is 87 a prime, we are actually asking if 87 has factors in **N** other than 1 and 87. Well, is 87 prime?
7. Which of the following are prime?

   a. 38 **c.** 51 **e.** 90 **g.** 75
   b. 49 **d.** 83 **f.** 109 **h.** 37

8. Explain why an even number greater than 2 cannot be prime.
9. Explain why a natural number greater than 5 and whose units digit is 5 cannot be prime.
10. In **N**, numbers that have factors other than 1 and themselves are called *composite numbers*. Is 1 a composite number? Then **N** can be broken into three subsets:

    (1) {composite numbers} (2) {prime numbers} (3) {1}

11. List the set of natural numbers that are factors of each of the following:

EXAMPLE

List the set of factors of 36.
Set of factors of 36 = {1,2,3,4,6,9,12,18,36}.
Certainly each of these is a factor of 36, because each will divide 36 exactly.

a. 10 **b.** 18 **c.** 25 **d.** 32 **e.** 48 **f.** 56 **g.** 72 **h.** 99

We have seen that a prime number has two and only two natural number factors, while composite numbers have more than two natural number factors.

1. List the *pairs* of factors for each of the numbers named in Exercise 11 on page 27.

EXAMPLE

List the pairs of factors of 36.
The set of pairs of factors of 36 = $\{1 \times 36,\ 2 \times 18,\ 3 \times 12,\ 4 \times 9,\ 6 \times 6\}$. Therefore, there are five pairs of factors of 36. The product of each pair is 36. Since 4×9 and 9×4 used the same factors, we shall not list them both. The order in which we write each pair of factors is not important.

Let us examine prime numbers in connection with addition.

2. How many pairs of natural numbers can you find such that the sum of each pair is 6? 8?
3. Is there a pair of prime numbers whose sum is 6? 8? 13? 17?
4. So far it may seem that we can find a pair of prime numbers that will add to give any even number. Have you been able to express the odd numbers above as the sum of two primes?
5. See if you can express all the even numbers from 20 to 36 as the sum of two primes.
6. Mathematicians have suspected that all even numbers can be expressed as the sum of two primes, but no one so far has been able to prove it. Find a pair of primes whose sum is 98; whose sum is 100.
7. We do know that every even number between 2 and 1,000,000 is the sum of two prime numbers. Do you think that this is proof that every even number is the sum of two primes?
8. Which prime number will never be one of the pair of different primes whose sum is an even number?
9. Can you find odd numbers that are the sum of two primes?
10. Find a pair of prime numbers that will add to the following:

a. 48	**c.** 22	**e.** 102
b. 64	**d.** 96	**f.** 500

11. Can you find two pairs of prime numbers each of which will add to the following? For example, $16 = 11 + 5$ and $16 = 13 + 3$.

a. 36	**b.** 60	**c.** 22	**d.** 44

1. A set with no members is called an *empty set.* At first thought, this may not appear useful, but mathematicians find the idea very useful. The number associated with an empty set is zero. Examples of the empty set are:

   a. The set of even numbers not divisible by 2
   b. The set of odd numbers divisible by 4 with no remainder
   c. The set of astronauts who have visited Mars

   Describe three other empty sets. The symbols commonly used to represent the empty set are

$$\{\ \} \quad \text{or} \quad \phi$$

2. Tabulate these sets, and state the number of members each set has.

   a. The set of even numbers between 1 and 10
   b. The set of months whose names begin with J
   c. The set of months with 32 days
   d. The set of months whose names end in "ber"
   e. The multiples of 4 between 1 and 21

3. The empty set is, by definition, a subset of every set. List all other subsets of the following set: set $T = \{a,b,c,d\}$.

4. While watching the football game between Glendale and Colton, Jane noticed that there were three sets of people on the field.

   a. The Glendale squad wore orange jerseys. There were 40 players on the squad, of whom 11 were on the first team. How many were not on the first team?
   b. The Colton squad had green jerseys. There were 32 players on the squad, including the starting 11. How many were not on the starting squad?
   c. The third set consisted of the three officials who wore striped shirts. How many people were on the field at any one time during the game?

5. Identify the element that does not belong in each set.

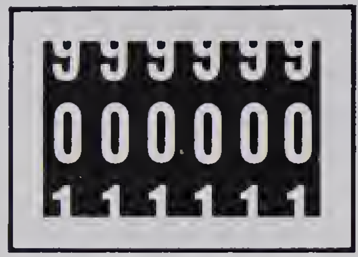
   a. The natural numbers less than 5: $A = \{0,1,2,3,4\}$
   b. Prime numbers less than 10: $C = \{1,3,5,7,9\}$
   c. Composite numbers less than 10: $\{2,4,6,8,9\}$

6. List the following sets of pupils in your class.

   set A = {girls} set B = {girls who ride bicycles}
   set C = {boys} set D = {boys who own dogs}
   set E = {pupils who walk to and from school}

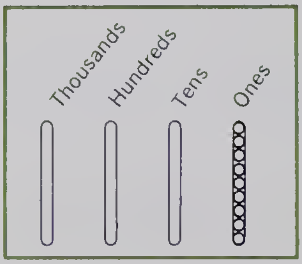
An important step in the development of the Hindu-Arabic system of numeration was taken when some forgotten mathematician introduced zero. But for notation and computation zero has several distinct but related uses.

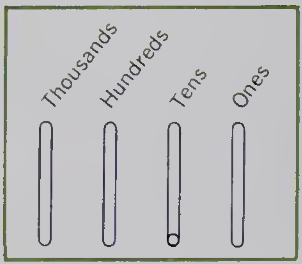
1. At the beginning of a trip Jim set the odometer of the car at zero. It looked like this:



Make a sketch to show how it looked after the car had traveled 10 miles; after 100 miles. Explain the meaning of zero in the Figure above and in each of your sketches.

2. If you write 7, and then place a zero after it, what number have you named? Explain how you have used the place value of the decimal system to multiply the value of the 7.
3. Show how you can multiply 7 by 1000.
4. The characteristics of the decimal system of numeration probably originated with the use of devices for computation such as the sand table, which we have already discussed.





Suppose each groove in the sand table is just long enough to hold 9 pebbles, as in the above Figures. If you wish to add one more pebble to the groove on the right to represent ten, you would remove the 9 pebbles from the ones groove, and place one pebble in the tens groove. What does the digit 1 tell you about the number of pebbles in one of the grooves? What does 0 tell you?

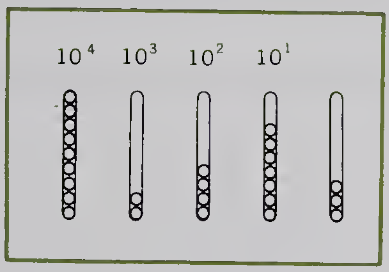
5. Make a sketch to represent each of the following numerals on a sand table:

a. 101 **c.** 1002 **e.** 3001
b. 3×10^2 **d.** 7×10^3 **f.** 2×10^4

6. Write each of the following numerals using exponents to indicate powers of 10.

a. 7000 **b.** 30 **c.** 40,000 **d.** 200

7. Using exponents to indicate powers of 10, analyze the value of each digit in the number represented in the sand table below.



8. Prepare a similar table to analyze 4005.

9. Zero, in a numeral, is often referred to as a "place holder." Actually, any digit is a place holder. Yet it took centuries to develop the idea of using zero to indicate "no pebbles" in a given groove on the sand table. Make a sketch to show the numeral 305 on the sand table. Then write the numeral in Greek, Egyptian, Roman, and Hindu-Arabic numerals. Which reflects more accurately what is on the sand table?

10. Another idea commonly associated with zero is its use as a *starting point* on a ruler, tape measure, or on a number line, as we have already seen. On the number line below, the distance from 0 to 6 is how many times the distance from 0 to 2?

11. Instead of a starting point, zero is often located near the middle of a scale, as on the globe in measuring north and south latitude, above and below sea level, or on a thermometer. In this case it is called a *point of reference.* Name two other uses of zero as a point of reference.

SPECIAL REPORT

The above Exercises contain several illustrations of different uses of zero as:

(1) zero as a "place holder"
(2) in writing numerals
(3) as a starting point or a reference point

Review the Exercises and give another illustration of each use.

At the right below is a *magic square*. The answers to these exercises will show you why it is "magic."

2	7	6
9	5	1
4	3	8

1. Adding vertically, what is the sum of the numbers named in each column?

2. What is the sum of the numbers named in each row?

3. What is the sum of the numbers named in each of the two *diagonals*?

4. Using the natural numbers from 1 through 9, see if you can construct a magic square different from the one in the diagram above.

5. If you can do the next nine exercises correctly, then you can make a magic square. Copy the square at the right, but instead of the letter, put in each cell the answer to the exercise below that corresponds with that letter.

A	B	C
D	E	F
G	H	I

a. $2^3 = ?$

b. The number of pints in 3 quarts if 2 pints make 1 quart

c. The number of hours you are awake each day if you sleep 8 hours

d. $2 \times 3 \times 3 = ?$

e. The number of dollars you would pay for 5 yards of cloth at $2 a yard

f. The number of dollars Jim must earn per hour if he is to earn $36 in 18 hours

g. $2^2 = ?$

h. The number of times Mike must run around the block if he wants to run 2800 yards and it is 200 yards around the block

i. The number of inches in a foot

6. Now check your square to see if it is magic. What is the sum of the numbers named in each row? What is the sum of the numbers named in each column? What is the sum of the numbers named in each diagonal? If they are not all the same, you have made a mistake. Find it and correct it.

SPECIAL PROJECT

People have been interested in magic squares for many centuries. There are many different kinds of magic squares. You can find much interesting information about them in various encyclopedias to which your teacher can refer you. Prepare a report for the class on some of the facts you consider most interesting.

Part One

1. Write each of the following as Roman numerals.

   a. 144 c. 49 e. 1966
   b. 68 d. 609 f. 2001

2. Write each of the following as numerals in the decimal system.

   a. XXXIV c. CV e. MDCXV
   b. MIV d. MCMII f. MCMLVI

3. Using exponents to indicate powers of 10, analyze each of the following to show what each digit means.

   a. 3,648 b. 256 c. 1000 d. 22,222

4. Each of the following is written in base 5. Using exponents to show powers of 5, analyze each numeral to show what each digit means.

   a. 23 b. 234 c. 300 d. 1,234

5. List the elements in the set of prime numbers less than 10.
6. Express each of the following as the product of prime numbers.

   a. 21 d. 56
   b. 34 e. 72
   c. 48 f. 78

7. Find 2 prime numbers whose sum is equal to each of the following:

   a. 46 d. 66
   b. 52 e. 88
   c. 58 f. 90

8. In the square at the right what numerals must be written in the cells containing letters in order to make a magic square?

   $a = ?$ $b = ?$ $c = ?$ $d = ?$

79	A	51
B	C	121
135	D	107

9. The magic total for the magic square at the right is 141. Copy the magic square and write the numerals in the cells containing letters in order to make it a complete magic square.

   $a = ?$ $b = ?$ $c = ?$ $d = ?$ $e = ?$ $f = ?$

A	B	C
D	E	65
F	11	56

Part Two

1. Given: set $B = \{2,4,6,8,10,12\}$
Which of the following are subsets of B?

   a. $\{8,10,12,14\}$ b. $\{1,2,3,4\}$ c. $\{2,4,6,8\}$

2. Which of the whole numbers is not in the set of natural numbers?
3. Using exponents to indicate powers of 10, analyze each of the following to show the value of each digit.

   a. 1,475,836 b. 70,080

In Problems 4, 5 and 6 consider:

set A = {natural numbers greater than 4 and less than 11}

4. List the elements of set A in proper notation.
5. Which of the following are not elements of set A?

   a. 4 c. 10 e. 0
   b. 5 d. 11 f. $6\frac{1}{2}$

6. List the elements in the following subsets of set A, using proper notation.

   a. set B = {odd numbers in set A}
   b. set C = {prime numbers in set A}
   c. set D = {numbers in set A that are divisible by 2}

7. How many elements are in each of the following sets?

   a. set V = {the most nickels Irene can get in changing a quarter}
   b. set W = {the even numbers between 1 and 11}
   c. set X = {the multiples of 5 that do not end in 5 or 0}

8. From set $Z = \{1,2,3,4,5,6,7,8,9\}$, select the element or elements that fit each of the following:

   a. The least prime number
   b. The number that is neither prime nor composite
   c. The smallest composite number
   d. The greatest composite number
   e. The prime factor or factors of the greatest even number
   f. The prime factor or factors of the largest odd number
   g. The prime numbers whose sum is equal to the least composite number in set Z
   h. The numbers which when multiplied by themselves will produce numbers which are also in set Z.

Part Three

1. Jim Stone, in telling about the vacation trip his family had taken to the western national parks, said that in returning to the eastern side of Glacier Park after visiting Lake McDonald on the western side, they had crossed the Continental Divide at Marias Pass at an altitude of 5216 feet. This is how much lower than the altitude of Yellowstone Lake, at 7000 feet?
2. On the drive from Glacier Park to Yellowstone Park the Stone family had stopped at Butte to visit the Anaconda Copper Refining plant, one of the world's largest. In the previous year 100,000 tons of copper, worth $62,000,000, had been refined. How much was the copper worth per ton?
3. It was 930 miles from the Black Hills to the Stone home in Chicago. They made the drive in 3 days. Jim said that to find the average number of miles driven per day you should divide the total number of miles by the number of days. What was the average number of miles driven per day?
4. On the second day after leaving the Black Hills the Stone family left Sioux Falls at 8:30 A.M., stopping a half hour for lunch at 12 noon. During the afternoon they stopped a half hour for a dish of ice cream. They arrived at Dubuque at 5:30 P.M. How long had they been driving, not including time out for lunch and refreshments?
5. The distance from Sioux Falls to Dubuque is 408 miles. What was the average number of miles driven per hour?
6. The Los Angeles Dodgers baseball team recently won the National League pennant. They played 162 games and won 98 of them. How many games did they lose?
7. The team that came in second in the league lost 68 games. All the teams in the league play the same number of games. How many games did the second-place team win?
8. The Dodgers won 44 more games than the last-place team. How many games did the last-place team win, and how many games did it lose?
9. The Dodgers won the World Series, receiving $255,750 from the game receipts. The money was divided into 31 equal shares. How much was each share?
10. The losing team also receives a part of the receipts from attendance at the World Series. This was $183,040, which was divided into 32 shares. How much was each share?

PROBLEM SOLVING

WORDS TO WATCH FOR

addend
arithmetic progression
bar graph
checking
closed sentence
conditional statement
equivalent
estimating
factor
horizontal scale
line graph
mathematical sentence
open sentence
pictograph
product
rounding
solve
statement
value
variable
vertical scale

The science of mathematics was developed as a means for solving the quantitative problems that arose in our physical, economic, and social environment. This use of problem solving is still an important aspect of mathematics.

Equally important, however, and often more interesting, is the exploration of mathematics as a science. For centuries many individuals, some known and some forgotten, representing many races, have been contributing to the growth and perfection of the science by developing the art of exploration and discovery to establish new relationships and new generalizations in various fields of mathematics.

You have already encountered several varieties of mathematical problems. One type includes the searching for new generalizations, such as the rules of operations. Another is the mathematical puzzle, such as the magic square or finding the missing digits in a division exercise. A third is a puzzle growing out of an application that is trivial or absurd, where the interest lies primarily in the mathematical relationships. An example of the third type is trying to measure out six quarts of water using 5-quart and 8-quart pails.

STEPS FOR MATHEMATICAL PROBLEM SOLVING

While there are no fixed rules for solving a mathematical problem, you can improve your ability by mastering the five simple steps that are illustrated with this problem:

PROBLEM

Find a simple rule for obtaining the product of two numbers which satisfy the following conditions.
(a) Each number is named by two digits.
(b) In each numeral the same digit is in the tens place.
(c) The numbers named by the units digits add to 10.

Step 1. *Be sure you understand the problem.*

List some factors illustrating the problem, as: 54×56, 62×68, etc.

The rule should be simple enough so that you can use it to find products without written computation.

Step 2. *Analyze the data.*

To provide some data for study, write down some products and see what generalizations are possible.

$14 \times 16 = 224$	$13 \times 17 = 221$
$24 \times 26 = 624$	$23 \times 27 = 621$
$34 \times 36 = 1224$	$33 \times 37 = 1221$

Step 3. *Discover new facts.*

Search for a uniform pattern in the data. For example, can you predict what the tens and units digits in the product will be, by looking at the units digits in the factors? If you think so, try it on a pair of other factors.

Can you find a clue to help you determine the digits preceding the tens digit in the product? Are these the product of the tens digits in the factors? If not, how are they related to the tens digits in the factors?

When you think you have a clue, try it out on a new pair of factors.

Step 4. *Follow up each promising lead, and verify it if true.*

If one of your clues suggests an idea that works on one pair of factors, try it on others. See if you can explain why it works.

Step 5. *Review your solution.*

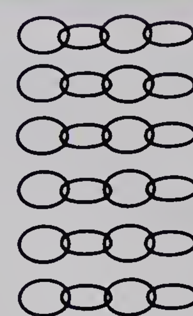
State your solution as a rule. Go over your work and see if you overlooked any clues. Was there an easier way to do it?

The mathematical puzzle often provides an interesting problem. Almost without exception such puzzles appear more difficult than they actually are. Follow the problem-solving steps carefully in the solution of such problems.

STEPS FOR SOLVING MATHEMATICAL PROBLEMS

1. Understand the problem.
2. Analyze the data.
3. Discover new facts.
4. Follow up and verify promising leads.
5. Review your solution.

PROBLEM



A farmer has six pieces of chain, each having four links. He would like to connect them into one chain. It costs 10¢ to cut a link, and 10¢ to weld a link together again. How can this be done most economically, and what will it cost?

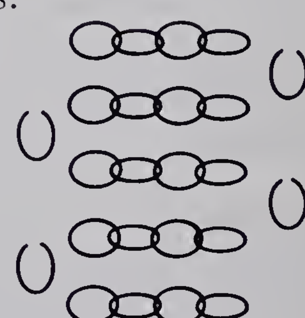
Step 1. *Do you understand the problem?*

Make a sketch of the six pieces of chain as shown above.

Step 2. *Study and analyze the data.*

What will it cost to connect them in the obvious manner? Do you see any other possible ways of making the chain?

Step 3. *Explore the data in order to discover new facts.*



You are looking for the procedure that will require cutting and welding the least number of links. One possibility is to cut all four links of one of the pieces. This has the advantage that there will be one less section to join up, and the links could be used as connections between the other sections.

Step 4. *Follow up each promising plan.*

Would it work? Would it cost less than the obvious method?

Step 5. *Review your solution.*

Are there any other possibilities? If so, test them.

Practice using each of the problem-solving steps on these mathematical puzzles.

1. Eric has 12 blue socks and 10 gray socks in the bureau drawer. He reaches in without looking to get a pair of the same color. How many socks must he take out to be sure he gets two of the same color?

2. If 48 players enter a tennis tournament, how many matches must be played to determine the singles champion?
   Note: This is an illustration of a problem where a diagram will be useful in helping you study the data to find clues. How many players were eliminated the first time they played? How many matches were played to eliminate them? Continue this line of questioning.

3. Mary received 1¢ on her first birthday, 2¢ on her second, 4¢ on her third, 8¢ on her fourth, and so on. On each birthday through her sixteenth she received twice as much as on her previous birthday. How much did she receive on her sixteenth birthday?
   Note: This is an illustration of a problem where a table will help you study the data.

Birthday Number	*Received*
1	1
2	$2 \times 1 = 2$
3	$2 \times 2 = 4$
4	$2 \times 2 \times 2 = 8$
5	$2^4 = 16$

   Remember! $2^4 = 2 \times 2 \times 2 \times 2$ The exponent 4 tells how many times 2 is used as a factor. What will be the exponent on her 6th birthday? Can you predict what it will be on her 16th birthday?

4. Jim is to cut a strip of cloth that is 25 yards long into 1-yard lengths. It takes him 1 minute to make 1 cut. How long will it take him to cut the 25 lengths?
   Note: Make a diagram. How many cuts must he make?

5. In a barnyard containing chickens and cows there are 36 heads and 100 feet. How many chickens are there, and how many cows?

6. A clock strikes on the hour, striking the number of the hour. How many times does it strike in one 24-hour day?

Nine

1. Perform the following multiplications.

   a. $9 \times 98{,}765{,}432$
   b. $9 \times 109{,}739{,}369$
   c. $9 \times 13{,}717{,}421$
   d. $9 \times 12{,}345{,}679$

2. Examine the products for 9 at the right. Copy them, and complete the table through 9×9. What is the sum of the numbers named by the digits in each product?

   $9 \times 1 = 9$
   $9 \times 2 = 18$
   $9 \times 3 = 27$

3. In which of the following is the sum of the numbers named by the digits equal to 9?

   a. 405 b. 621 c. 3113 d. 2063 e. 756

   In d: $2 + 0 + 6 + 3 = 11$ Add again: $1 + 1 = 2$
   In e: $7 + 5 + 6 = 18$ Add again: $1 + 8 = 9$

4. Which of the above (Exercise 3) are exactly divisible by 9? Or 9 is a factor of which of the numbers named in Exercise 3?

5. Copy and complete this table after dividing each number by 9. If the sum of the numbers named by the digits is a 2-digit numeral, add again as in Exercises 3d and 3e above.

	Sum	*Quotient*	*Remainder*
504	?	?	?
627	?	?	?
3114	?	?	?
1721	?	?	?
724	?	?	?

6. Examine your table and write the answer to this question: How can you tell without dividing what the remainder will be when you divide a number by 9?

7. It appears that if the sum of the numbers named by the digits in a number is 9, the number is exactly divisible by 9 (no remainder). Test this conclusion with several 3-digit and 4-digit numerals.

8. We know that 513 is divisible by 9. Is 315 divisible by 9? Does changing the order of digits in a numeral affect its divisibility by 9? Test your answer with several other numbers.

Suppose you are adding 28 and 67. Dividing 28 by 9 shows that 28 is made up of 3 nines and a remainder of 1. Similarly, 67 is made up of 7 nines and a remainder of 4. Therefore, the sum of 28 and 67 will be 10 nines and a remainder of five.

28	3 nines + 1	Excess = 1
+ 67	7 nines + 4	Excess = 4
95	10 nines + 5	Excess = 5

Casting out nines from the sum, 95, we find that the excess (remainder) is still 5.

$$9 + 5 = 14; \qquad 1 + 4 = 5$$

This illustrates the fact that you get the same excess whether you add first and then cast out the nines, or cast out the nines first and then add the excesses. This discovery provides a simple way of checking answers in addition:

(1) Cast out nines from each addend.
(2) Add the excesses, casting out nines again if necessary.

The result should be the same as when you cast out nines from the sum.

This check is not foolproof. If you *transpose* the digits in the sum, writing 59 instead of 95, it would still check. If the excesses do not agree, your answer is *surely wrong*. If they agree it is *probably right*.

EXAMPLE

Add:	*Casting out nines*	*Excesses*
452	4 + 5 + 2 = 11; 1 + 1 = 2	2
507	5 + 0 + 7 = 12; 1 + 2 = 3	3
819	8 + 1 + 9 = 18; 8 + 1 = 0	0
763	7 + 6 + 3 = 16; 1 + 6 = 7	7
2541	2 + 5 + 4 + 1 = 12; 1 + 2 = 3	12, 1 + 2 = 3

The sum of excesses is the same when the nines are cast out from the sum as when they are cast out from each addend and then added.

Add: Check your answer by casting out nines.

1.	**2.**	**3.**	**4.**	**5.**
75	472	809	15,421	11,031
49	531	43	7,958	6
86	109	511	18,399	89,405
70	56	796	7,075	7,602

OTHER OPERATIONS AND CASTING OUT NINES

The process of casting out nines is equally useful for checking answers in subtraction. Let us see if the result will be the same whether you cast out nines and then subtract, or subtract, and then cast out nines in the difference.

EXAMPLE

Subtract:

		Casting out nines	*Excesses*
1021	minuend	$1+0+2+1=4;\ 4+9=13$	13
-735	$-$ subtrahend	$7+3+5=15;\ 1+5=6$	-6
286	difference	$2+8+6=16;\ 1+6=7 \longleftrightarrow$	7

Note that 9 must be added to the excess in the minuend whenever the excess is smaller than that in the subtrahend.

Subtract, and check by casting out nines.

1. $\begin{array}{r} 732 \\ -341 \\ \hline \end{array}$ **3.** $\begin{array}{r} 1856 \\ -907 \\ \hline \end{array}$ **5.** $\begin{array}{r} 10{,}915 \\ -8{,}575 \\ \hline \end{array}$ **7.** $\begin{array}{r} 98{,}732 \\ -67{,}775 \\ \hline \end{array}$

2. $\begin{array}{r} 858 \\ -507 \\ \hline \end{array}$ **4.** $\begin{array}{r} 7285 \\ -6591 \\ \hline \end{array}$ **6.** $\begin{array}{r} 7325 \\ -6517 \\ \hline \end{array}$ **8.** $\begin{array}{r} 41{,}351 \\ -35{,}065 \\ \hline \end{array}$

In checking a multiplication exercise, the excess of nines is found for each of the two factors. The product of these two excesses should equal the excess of nines in the product of the factors.

EXAMPLE

Multiply:

		Casting out nines	*Excesses*	
482	factor	$4+8+2=14;\ 1+4=5$	5	
$\times 796$	$\times$ factor	$7+9+6=22;\ 2+2=4$	$\times 4$	
2892	partial products		20;	$2+0=2$
4338				$\updownarrow$
3374				
383,672	product	$3+8+3+6+7+2=29;\ 2+9=11;$		$1+1=2$

Find the products, and check by casting out nines.

9. 826×819 **12.** 865×48 **15.** 8765×903
10. 6321×46 **13.** 598×54 **16.** 7241×369
11. 1345×6 **14.** 364×274 **17.** 339×39

You know that you can check your answer in a division exercise by using the relationship:

$$\text{divisor} \times \text{quotient} + \text{remainder} = \text{dividend}$$

which can be stated as

$$\text{factor} \times \text{factor} + \text{remainder} = \text{product}$$

Thus, if you divide 89 by 16,

$$89 \div 16 = 5, \text{ remainder } 9$$

and you wish to check your answer, your check would be:

$$(16 \times 5) + 9 = 89$$

Note that parentheses are placed around the factors 16 and 5 so that you will *first* multiply and *then* add.

Instead of using the original numbers, you could use the excess of nines for each number:

		Excesses
factor	16	7
factor	5	5
remainder	9	0

(factor × factor) + remainder
$(7 \times 5) + 0 = 35;\ \ 3 + 5 = 8$

EXAMPLE

Divide: $59{,}323 \div 46$

```
     1,289
46)59,323
   46
   133
    92
    412
    368
     443
     414
      29  = remainder
```

		Excesses
factor	46	1
factor	1289	2
remainder	29	2
product	59,323	4

Check:

$(1 \times 2) + 2 = 4$
(factor × factor) + remainder = product

Divide, and check by casting out nines.

18. $6932 \div 13$ **20.** $9919 \div 17$ **22.** $72{,}393 \div 36$
19. $30{,}117 \div 9$ **21.** $684 \div 19$ **23.** $20{,}425 \div 25$

Do each of the following exercises and check your answer by casting out nines.

A. *Add:*

1. 11,236
8,472
43,005
17,756

2. 14,325
99,818
19,226
303,039

3. 809
13
303
4
155
96

4. 12,454
8,308
41,756
107

5. 3,063
19,967
31,365
87,565

6. 726
50
9
869
983
684

7. 1,798
81,375
503
90,545

8. 37
343
87,549
98,587

9. 657
58
982
76
7,859
56,785

B. *Subtract:*

1. 8705
− 1906

2. 704
− 309

3. 536
− 248

4. 671
− 172

5. 46,012
− 9,678

6. 90,375
− 70,079

C. *Multiply:*

1. 151×26
2. 138×995
3. 885×175
4. 378×155
5. 437×375
6. 476×75
7. 917×88
8. 246×31
9. 739×54

D. *Divide:*

1. $65{,}087 \div 65$
2. $7365 \div 841$
3. $8764 \div 123$
4. $60{,}522 \div 11$
5. $4357 \div 73$
6. $5460 \div 25$
7. $30{,}426 \div 15$
8. $2029 \div 20$
9. $18{,}148 \div 132$

A *mathematical sentence* is a statement of a mathematical relationship. Sentences Ia and Ib express the same idea.

I *a.* The sum of three and four is seven.
b. $3 + 4 = 7$

Ib is more concise because it uses numerals and mathematical symbols. Sentences IIa and IIb are also expressions of the same idea, with IIb using the symbol $>$ to express "is greater than."

II *a.* Eight is greater than five.
b. $8 > 5$

In sentence IIIb the symbol $<$ is used to express "is less than," so the sentence expresses the same idea as IIIa.

III *a.* Eighteen is less than twenty-five.
b. $18 < 25$

Sentences I, II, and III are all examples of "true statements." Sentences IVa and IVb are examples of "false statements." Nevertheless, true or false, they are still sentences.

IV *a.* Eleven is greater than thirty.
b. $11 > 30$

How could you change the symbol between 11 and 30 to make IVb a true statement?

Mathematical sentences that express relationships which are either true or false are *closed sentences.*

1. Examine sentences I, II, III, and IV. Are they closed sentences?

2. Make sentence Ib a false statement by replacing one of the numerals with another numeral.

3. Make sentence IIb a false statement by replacing the symbol between the numerals with another symbol.

4. Using the numerals in IIIb, write a false statement.

5. In the following sentences, ○ indicates where a symbol is to be supplied. Write each sentence as a true statement, using $>$, $=$, or $<$ as called for.

a. $5 + 8 \bigcirc 13$
b. $7 + 16 \bigcirc 25$
c. $45 - 19 \bigcirc 25$
d. $9 + 7 \bigcirc 16$
e. $18 + 13 \bigcirc 30$
f. $15 + 18 \bigcirc 33$
g. $48 \div 12 \bigcirc 4$
h. $5 \times 9 \bigcirc 47$
i. $32 - 19 \bigcirc 13$
j. $4 \times 12 \bigcirc 3 \times 15$

TRANSLATING VERBAL SENTENCES

An *open mathematical sentence* is a mathematical sentence in which a letter is used to hold the place of a number. A letter used in this way is called a *variable*. An open sentence is neither true nor false. If a number is put in place of the variable, the sentence will be either true or false. A verbal sentence about numbers can be translated into an open sentence if a variable is used in place of some number.

EXAMPLES

1. *a.* The sum of some number and 17 is 23.
b. $n + 17 = 23$

Both of these sentences express the same idea. Notice in sentence b that n represents "some number." In other words, n holds the place of the number. (You may have already guessed what this number is.)

2. *a.* The product of 4 and some number is 36.
b. $4 \times y = 36$ or $4y = 36$

Notice that we use y to hold the place of "some number." We can use any letter to represent "some number." The open sentence $4 \times y = 36$ can be shortened to $4y = 36$, as $4y$ means 4 times y.

Translate each of the following sentences into open mathematical sentences, using mathematical symbols.

1. One-third of some number is greater than 9.
2. If 8 is added to some number the sum is less than 20.
3. If 5 is subtracted from some number the result is 15.
4. Some number divided by 5 is equal to 3.
5. 48 is 8 times some number.
6. Three-fourths of a certain number is less than 15.
7. If some number is added to 18 the result is 26.
8. Four times some number is more than 15.
9. Five-eighths of some number is greater than 25.
10. If 6 is added to some number the sum is equal to 20.
11. 25 is four less than some number.
12. 18 is 3 more than some number.
13. Two-thirds of some number is less than 19.
14. Some number divided by two is 10.
15. If twelve is subtracted from some number the result is 8.

It is also possible to write English sentences which describe in words what open mathematical sentences say in symbols. Study the following examples carefully.

EXAMPLES

1. *a.* $\frac{1}{2} \times m > 12$ or $\frac{1}{2}m > 12$
   b. One-half of some number is greater than 12.

Notice that the open sentence $\frac{1}{2} \times m > 12$ was shortened to $\frac{1}{2}m > 12$ since $\frac{1}{2}m = \frac{1}{2} \times m$.

2. *a.* $n \div 9 = 8$
   b. Some number divided by 9 is eight.

Translate each of the following open sentences into English sentences. Refer to the Examples above.

1. $17 - n = 12$
2. $15 + x = 27$
3. $18 + 13 > y$
4. $29 - 12 < n$
5. $2b = 18$
6. $\frac{1}{2}n = 6$
7. $4a > 19$
8. $12 = 5 + n$
9. $4b = 16$
10. $43 > y - 15$
11. $16 - a = 9$
12. $\frac{1}{3}a = 9$
13. $100 - n = 81$
14. $x - 49 > 50$
15. $31 - 16 < n$
16. $16 + n > 25$
17. $32 - y = 19$
18. $x + 12 = 22$
19. $a - 9 < 17$
20. $56 \div n = 14$
21. $x \div 4 = 5$
22. $15 + 12 < y$
23. $15 = 6 + n$
24. $36 \div x = 9$
25. $3a = 39$
26. $\frac{1}{2}y = 20$
27. $4 + 2a = 10$
28. $31 - 9 < n$
29. $3a > 15$
30. $\frac{1}{3}x = 6$
31. $12 \div x = 3$
32. $6y = 30$
33. $13 + 5 > n$
34. $a - 4 = 18$
35. $27 > 9b$
36. $7x = 21$
37. $12 - 5 < n$
38. $\frac{1}{2}b = 16$
39. $4 - a = 1$
40. $6y + 5 = 29$

A mathematical statement written with an equal sign is an *equation.* An equation states that the symbols on both sides of the equal sign name the same number. An equation may be a true statement, a false statement, or a *conditional* statement (open sentence). To *solve* a conditional statement you must find the number or numbers to replace the variable so that the equation makes a true statement.

Often you can solve an equation by using the powerful tool of mathematics called the "if–then" type of reasoning. That is, you examine the conditions and can say, "If this is true, then that must be true also." For example, many equations are about addition, with one of the terms missing. Here is an equation about addition:

$$\underset{\text{addend}}{8} + \underset{\text{addend}}{15} = \underset{\text{sum}}{23}$$

If s represents the sum and a and b represent the addends, then:

$$a + b = s$$

You know also that you can find either addend if you subtract the other from the sum; that is

$$a = s - b \qquad \text{or} \qquad b = s - a$$

1. Write the equation $8 + 15 = 23$ in each of the two forms just shown. Are the statements true?

2. You can also write conditional equations about addition in each of the three equivalent ways. Thus:

$$\begin{aligned} \text{If:} \quad & 8 + x = 23 \\ \text{then:} \quad & 8 = 23 - x \end{aligned}$$

Write the equation in the third form. You can see that when you have the variable alone on one side, this form leads you directly to the solution.

3. The three equations are called *equivalent* because all have the same solution. Find the value for x that makes the third equation a true statement. Does it make the other two statements true also?

4. Given the conditional statement:

$$8 + 15 = n$$

a. Which of the terms is missing, the sum or an addend?

b. Complete the following statement in two ways.
If $8 + 15 = n$, then _?_.

c. Replace n with a value that makes the statement true.

5. Given the equation:

$$x - 25 = 13$$

a. What term is missing, the sum or an addend?
b. Complete the following statement in two ways.
If $x - 25 = 13$, then __?__.
c. Replace x with a value that makes the statement true.

6. Which term is missing in this equation?

$$35 - y = 15$$

Write the equivalent equation that gives you the solution directly. Solve the equation.

7. Which term is missing in this equation?

$$28 - 15 = n$$

Will an equivalent equation give you a more direct solution? Find the value for n that makes the statement true.

Write the equivalent equation, where necessary, that gives you the direct solution for each of these statements. Then find the value of the variable. Check your answer by replacing the variable in the original sentence to see if it is true.

8. $y + 17 = 25$
9. $16 + 9 = a$
10. $x - 16 = 19$
11. $48 - n = 27$
12. $47 - 33 = y$
13. $48 - a = 41$
14. $29 - n = 16$
15. $x - 15 = 33$
16. $33 - 18 = n$
17. $a - 25 = 15$
18. $n + 29 = 37$
19. $15 + 27 = y$
20. $27 - y = 15$
21. $8 + n = 34$
22. $b + 16 = 50$
23. $16 + 5 = n$
24. $14 + 11 + 21 = x$
25. $33 - n = 19$
26. $56 = 23 + b$
27. $n = 13 + 17$
28. $17 - x = 9$
29. $13 + y = 30$
30. $17 + n = 25$
31. $x + 24 = 36$
32. $b + 35 = 42$
33. $14 + x = 38$
34. $13 - 7 = x$
35. $4 + y = 18$
36. $16 = a + 5$
37. $28 - b = 15$
38. $n + 32 = 47$
39. $x - 12 = 7$
40. $16 + 9 + 7 = b$
41. $23 = x - 15$
42. $61 = 73 - y$
43. $a + 3 + 5 = 11$
44. $13 = n + 4$
45. $22 = 14 + y$

Frequently you will recognize that an equation describes a multiplication exercise. For example:

$$\underset{\text{factor}}{9} \times \underset{\text{factor}}{17} = \underset{\text{product}}{153}$$

If p represents the product and x and y are the factors, this statement says:

$$x \times y = p$$

If either factor is unknown, find it by dividing the product by the other.

$$x = p \div y \qquad \text{or} \qquad y = p \div x$$

1. Write the equation, $9 \times 17 = 153$, in each of the two forms just shown. Are they true statements?

2. You can also write an equation containing a variable in each of the three equivalent forms; thus

$$\begin{aligned} \text{If: } & n \times 17 = 153 \\ \text{then: } & 17 = 153 \div n \end{aligned}$$

Write the equation in the third equivalent form. Which form leads most directly to the solution? Why? Then $n = ?$

3. Given the equation: $9y = 153$

a. Which of the terms is missing, the product or one of the factors?
b. Complete the following statement in two ways.
If $9y = 153$, then $\underline{\ ?\ }$.
c. What is the solution?

4. Given the equation: $360 \div 15 = x$

a. Which of the terms is missing, the product or one of the factors?
b. Complete the following statement in two ways.
If $360 \div 15 = x$, then $\underline{\ ?\ }$.
c. Which of the equivalent forms leads most directly to the solution? Why?
d. What is the solution?

Find the value of the variable in each of the following conditional statements so as to make each a true statement.

5. $13n = 91$
6. $19a = 152$
7. $n \div 14 = 49$
8. $686 \div n = 14$
9. $375 \div x = 15$
10. $a \div 17 = 9$
11. $49y = 343$
12. $b \div 23 = 13$
13. $15 \div y = 135$
14. $27x = 324$

Translate each of these sentences into mathematical sentences using mathematical symbols and representing "a certain number" by whatever variable you choose. Then solve to find the value for the variable that makes the statement true. Check your answer by replacing the variable in the original sentence to see if it is true.

EXAMPLE

54 times a certain number is 810.
Let n represent the missing number.

Then	$54n = 810$	$x \times y = p$
	$n = 810 \div 54$	$y = p \div x$
	$n = 15$	

Therefore, the number is 15.
Check: $54 \times 15 = 810$

1. If 14 is added to a certain number the sum is 33.
2. Fifteen times a certain number is 75.
3. If 34 is added to a certain number the sum is 51.
4. If 17 is subtracted from a certain number the difference is 19.
5. If 180 is divided by a certain number the quotient is 15.
6. If 9 is multiplied by a certain number the product is 135.
7. If a certain number is added to 39 the result is 81.
8. Fifty-four times a certain number is 486.
9. Seventeen less than a certain number is 48.
10. If a certain number is divided by 17 the quotient is 11.
11. If a certain number is subtracted from 52 the result is 27.
12. If 27 is multiplied by a certain number the product is 621.
13. If a certain number is added to 17 the sum is 41
14. If 1875 is divided by a certain number the quotient is 25.
15. If a certain number is divided by 25 the quotient is 17.
16. If a certain number is subtracted from 41 the result is 17.
17. If a certain number is added to 42 the sum is 99.
18. If a certain number is divided by 19 the quotient is 15.
19. Twenty-five times a certain number is 175.
20. If a certain number is subtracted from 56 the result is 29.
21. If 360 is divided by a certain number the result is 24.
22. Seventeen times a certain number is 136.

Jane spent a few weeks in Hawaii last winter with her parents. They had traveled by air from San Francisco to Honolulu. "It was exciting," Jane said, "to fly across 2400 miles of ocean."

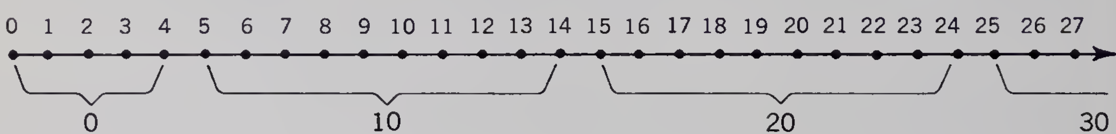
The distance from San Francisco to Hawaii is listed in travel folders as 2410 miles. Jane changed this to 2400 miles because it is easier to think of 2400 than 2410 and because greater precision is unnecessary. This *rounding of numbers* is frequently desirable when precision is not required.

Jane's method of rounding was informal, and it is not easy to see what rule, if any, she was following. There are, however, some simple rules that help to make the process of rounding systematic.

1. In rounding to the nearest tens digit, look at the digit in the ones place of the numeral. If it is 5, 6, 7, 8, or 9, then
(1) replace the tens digit with the next greater digit and
(2) replace the ones digit with a zero.

However, if the digit in the ones place is 4, 3, 2, or 1 then
(1) leave the tens digit as it is and
(2) replace the ones digit with zero.

Thus 24 is rounded *down* to 20, while 27 is rounded *up* to 30. The following diagram illustrates rounding to the nearest 10. In the diagram, where should the brace marked "30" end?



State whether each of the following would be rounded down or up to the nearest ten.

a. 93 **b.** 563 **c.** 759 **d.** 1632 **e.** 27,653 **f.** 19,995

2. Round each of the above (Exercise 1) to the nearest ten.

3. In rounding to the nearest ten:
Any number from 75 through 79 will be rounded to _?_.
Any number from 71 through 74 will be rounded to _?_.
HINT: Use the illustration in Exercise 1.

4. If you are to round a number to the nearest hundred, the key is the digit in the tens place of the numeral. If the value of the digit is 5 or greater than 5, then replace the hundreds digit with the next greater digit, placing zeros in both the tens and ones places. If the value of the tens digit is less than 5, leave the hundreds digit as it is and place zeros in both the tens and ones places.

Round each of the following to the nearest hundred.

a. 419 **b.** 747 **c.** 666 **d.** 1988 **e.** 945 **f.** 7635

5. State a rule for rounding to the nearest thousand; to the nearest ten thousand. Round each of these to the nearest thousand; to the nearest ten thousand.

a. 62,743 **b.** 49,545 **c.** 83,449 **d.** 28,721

6. In broadcasting a football game a TV announcer said that there were about 38,000 people in the stadium. The paid attendance was later reported as 37,583, so the announcer had rounded the number to the nearest thousand. Suppose he had said the attendance was about 40,000. He would have been rounding the figure to the nearest _?_.

In problems that deal with money, we use the same procedure. For example, $1.56 is closer to $2.00 than to $1.00. Similarly, $8.47 is closer to $8 than to $9. If we have something like $27.50, we are just as close to $28 as we are to $27. In this case we round up. Thus $27.50 rounded to the nearest dollar is $28. Rounding $27.50 to the nearest ten dollars, we have $30.

In rounding to the nearest dime, use the same procedure. For example, $1.56 is closer to $1.60 than to $1.50. Also, $1.54 is closer to $1.50. Finally, $1.55 would be rounded up to $1.60. In short, it is the digit in the cents position that determines whether you round up or down in the dimes position.

7. Round each of the following to the nearest dollar.

a. $139.62 **b.** $2463.49 **c.** $199.51 **d.** $114.25

8. Round each of the amounts in Exercise 7 to the nearest hundred dollars.

9. Round each of the amounts in Exercise 7 to the nearest ten dollars.

10. Round each of the amounts in Exercise 7 to the nearest dime.

11. When items are listed in a grocery store at 2 for 29¢, you are charged 15¢ for one item. Why?

12. In fact, when items are listed at 3 for 29¢, you would no doubt have to pay 10¢ for one item. Does this practice follow the rule for rounding off numbers?

Many problems encountered in everyday activities arise out of the *applications* of mathematics. The solution of such problems requires several steps in addition to those used in solving mathematical problems. This is because you must first identify the mathematical elements in the problem situation, using only those that are relevant, and decide what computations are needed. A set of steps that provides a systematic problem-solving approach is illustrated below.

STEPS FOR SOLVING APPLIED PROBLEMS

1. Understand the problem.	**2.** Note what the problem asks for.	**3.** Look for hidden questions.
6. Check your answer.	**5.** Set up and solve the conditional statement(s).	**4.** Estimate a reasonable answer.

PROBLEM

George is saving money to purchase a bicycle. His father will give him 50 cents to add to each dollar he earns. Last month he earned \$13.25 working at a filling station, \$4.75 washing cars, and \$2.00 mowing lawns. How much will he have altogether, with what his father gives him?

Step 1. *Be sure you understand the problem.*

Put yourself in George's place and see if you understand his problem. State clearly to yourself what information you have. Sometimes you may have to look in the glossary or the dictionary to find the meanings of some words.

Step 2. *Note what the problem asks for.*

Do not find the answer to the wrong question. Note that in this problem you are to find what George will have altogether.

Step 3. *Look for "hidden" questions.*

In some problems you find all the data you need; in others you may have to make some calculations before going ahead. In this problem you have to answer two questions before reaching the final answer:

(1) How much was the total of George's earnings?

(2) How much did his father give him?

Always state and answer each of the hidden questions as you find them.

Step 4. *Make an estimate of a reasonable answer.*

A useful procedure is to use rounded numbers so that you will not require calculations that call for pencil and paper. After rounding George's earnings to the nearest \$1, they are:

$$\$13 + \$5 + \$2 = \$20$$

Therefore, his father will give him about \$10. The total is about \$30.

Step 5. *Set up the conditional statement, and then find the missing value.*

In this problem you need to add to find how much George earned, divide (by 2) to find what his father gave him, and then add to find what he will have altogether. Set up a conditional statement (or statements) to show what operations are to be performed, using n or some other letter to indicate what numbers are to be found. Each hidden question calls for a separate conditional statement. It is a good idea to omit dollar signs and other labels, so that the conditional statement directs your attention solely to the numbers and the operations which are to be performed.

Thus, the conditional statement which will enable you to find George's earnings is:

$$13.25 + 4.75 + 2.00 = n$$

Then $$n = 20.00$$

To find the total George will have, we have to add what his father gave him, which is half of his earnings:

$$20 + (20 \div 2) = T$$
$$T = 30.00$$

George will have a total of \$30.00.

Note: Parentheses are placed around $20 \div 2$ so that we will divide first and *then* add. The parentheses will help to prevent you from mistakenly adding the 20 and 20 and then dividing by 2.

Step 6. *Check your answer.*

Compare your computed answer with your estimate from Step 4. If the two are not reasonably close, review your computations. (In this problem, both happen to be the same.)

Problems which arise out of applications of mathematics often seem much more difficult than they actually are. The process of "translating" a problem into a conditional statement and the mathematics necessary to solve it is simplified by following a definite pattern such as the one illustrated above. You will have success with this type of problem if you study carefully the steps outlined above and if you apply these steps to each problem that you have to solve.

The following problems will give you practice in using the steps for solving applied problems.

1. On a trip of 600 miles Mr. Fraser's car averaged 15 miles per gallon of gasoline. He paid 31.9¢ per gallon of gasoline. How much did the gasoline cost him?

a. Do you understand the problem? Put yourself in Mr. Fraser's place, and explain the situation in your own words. What does "averaged 15 miles per gallon" mean?
b. State precisely what you are to find.
c. What other questions must you answer before you can find it?
d. Make an estimate of a reasonable answer, using 30¢ per gallon as the price of gasoline.
e. Set up the conditional statements to express these questions and to solve the stated problem.
f. Solve the conditional statements.
g. Compare your computed answer with your estimated answer.

2. Mike plans to build a bookshelf for his study. He purchased the following supplies: boards, \$6.50; nails and hardware, \$1.10; paint, 95¢. How much change should he receive from a \$10 bill?

a. What are you asked to find?
b. What other questions must you answer first?
c. Make an estimate of a reasonable answer. Round \$6.50 to the nearest dollar. Round \$1.10 to the nearest dollar. Round 95¢ to the nearest dollar.
d. Set up and solve the conditional statements.
e. Compare your computed and estimated answers.

3. Mr. Atchison's orchard produced 3200 bushels of apples last year. He sold them at \$2.75 per bushel. Costs of labor and other expenses amounted to \$4300. How much was earned from the orchard after these expenses were met?

a. Do you understand the problem? Imagine you are in Mr. Atchison's place, and explain it in your own words.
b. What are you to find?
c. What questions must you answer before you can find it?
d. Estimate a reasonable answer: suppose 3000 bushels were sold at \$3 per bushel, and his expenses were \$4500.
e. Set up and solve the conditional statements.
f. Compare your estimated and calculated answers.

Practice using each problem-solving step on each problem.

1. Jim works at the soda fountain in a drug store at 85 cents an hour. He works 4 hours each day after school, 5 days a week. How much does he earn each week?

2. Last summer, George worked in a canning factory at $50 a week. He worked 8 hours a day, 5 days a week. What did he earn per hour?

3. After the first month, George's wages were increased by 10 cents an hour. How much was the increase in earnings per week?

4. Henry works as a gas station attendant for 4 hours a day after school 5 days a week. On Saturdays he works for 8 hours. He is paid $1.15 an hour. How much does he earn on each school day?

5. How much does Henry earn each week?

6. Helen takes subscriptions to a magazine. For each $2.50 subscription she earns 50 cents. Last week she got 18 subscriptions. How much did she earn?

7. Sarah takes care of the chickens on a farm and earns money from selling the eggs. Last week she collected $11.15, $13.99, $9.75, and $10.40 on successive days for selling eggs. How much did she collect last week?

8. Sarah had various expenses that totaled $32.40 for last week. How much remained from her earnings for selling eggs?

9. Fred delivers advertising circulars every Saturday, working 8 hours at $1.20 an hour. What does he earn each Saturday?

10. Mike sells newspapers each afternoon and receives 25 cents for each $1 worth of newspapers he sells. One afternoon he sold $7 worth of newspapers. How much did he earn?

11. Mildred does secretarial work each Saturday at a doctor's office in her neighborhood. She works 8 hours at $1.30 an hour. How much does she earn each Saturday?

12. Mildred works 50 Saturdays each year. What does she earn each year?

13. Last summer, Milton had a job picking apples and was paid 15 cents a bushel. One week he picked 480 bushels. How much did he earn?

14. Mary earns money baby-sitting. She charges 50¢ for each hour before midnight and 75¢ for each hour after midnight. If she arrived at the Allen's house at 8:30 P.M. and left for home at 2:00 A.M., how much money did she earn?

The use of rounded numbers is very convenient in estimating a reasonable answer in problem solving. For many problems such an estimate gives you an answer that is as precise as you need.

On a sheet of paper list the numerals 1 to 8. After each numeral place the letter to indicate the answer you choose, after you have made an estimate with rounded numbers. State whether you think your answer will be a little too large or too small. Then solve the problem and see how near your estimate came to the exact answer.

1. What will 21 pounds of sugar cost at 14 cents per pound? (Round 21 to 20.)

   a. \$1.98 **b.** \$2.28 **c.** \$2.80 **d.** \$3.20

2. How long will it take Arthur's father to drive to Centerville, 387 miles away, if he averages 40 miles per hour? (Round 387 to 400.)

   a. 8 hours **b.** 10 hours **c.** 12 hours **d.** 14 hours

3. Mary read 25 pages of a novel in 45 minutes. How many minutes will it take her to read the entire novel if it contains 420 pages?

   a. 840 **b.** 600 **c.** 250 **d.** 200

4. John works 9 hours a week after school taking care of lawns at 85 cents an hour. How much does he earn per week?

   a. \$6 **b.** \$7 **c.** \$8.50 **d.** \$9.50

5. The flight from San Francisco to Hawaii is 2410 miles and takes $4\frac{1}{2}$ hours. What speed does the plane average per hour?

   a. 380 m.p.h. **b.** 480 m.p.h. **c.** 550 m.p.h. **d.** 660 m.p.h.

6. Helen's class is planning to visit the museum. They can hire a bus for \$8. There are 37 in the class. How much will each have to pay?

   a. 15 cents **b.** 20 cents **c.** 25 cents **d.** 30 cents

7. Margaret bought a pair of shoes for \$6.85 and rubbers for \$2.48. How much change should she receive from a \$10 bill?

   a. \$2.00 **b.** \$1.75 **c.** \$1.50 **d.** 50 cents

8. Ida bought 16 yards of ribbon at 22¢ a yard. She gave the clerk a \$5 bill. How much change should she receive?

   a. \$4.50 **b.** \$3.75 **c.** \$2.80 **d.** \$1.50

In working each problem be sure to use each of the problem-solving steps. When your problems have been scored, examine each mistake to see which of the steps have been incorrectly used.

1. Henry works after school at a lunch counter, two hours a day, five days a week. He is paid $1.75 an hour. How much does he earn each week?
2. Jim bought a pair of hunting boots for $14.75, and gloves for $2.75. How much change should he receive from a $20 bill?
3. When the Millers drove from San Jose to Portland their car used 52 gallons of gasoline. The distance is 780 miles. How many miles per gallon of gasoline did the car average?
4. Mr. Edwards started out on a trip of 390 miles with the tank full of gasoline. After driving 135 miles he found that it took 9 gallons to refill the tank. At this rate how much gasoline will he use on the trip?
5. A moving van is transporting furniture to a city 1350 miles away. The driver intends to make the trip in 3 days. How many miles per day must he average?
6. A bus averages 46 miles per hour. How long will it take the bus to travel from Minneapolis to St. Louis, a distance of 598 miles?
7. George plans to purchase a bicycle that will cost $56. He can earn $1.75 an hour working at the supermarket. How many hours will he need to work to pay for the bicycle?
8. Helen works as cashier at a theater 32 hours a week at $1.65 an hour. Edith works as a secretary 26 hours a week at $2 an hour. Which girl earns more per week, and how much more?
9. Jim purchased a boat for $230, paying $50 in cash, and the remainder in 12 equal monthly payments. How much was each payment?
10. The bus fare between Oakdale and Clear Lake is 95¢. The bus company sells a 10-trip ticket for $8. If you are to make 10 trips, how much per trip do you save by purchasing a 10-trip ticket instead of paying separately for each trip?
11. The one-way train fare from New York to Washington, D.C. is $10.85, and the round-trip fare is $21.00. How much money will the Nelsons save if they buy two round-trip tickets instead of four one-way tickets?
12. Joe earns $1.25 an hour. He works 7 hours a day, 5 days a week. What are his weekly earnings?

FIND THE HIDDEN QUESTION

In many of these problems you must first answer a hidden question before you can answer the question stated in the problem. In each, first state and answer the hidden question; then solve the problem.

EXAMPLE

Mrs. Wagner bought 5 pounds of tomatoes at 12 cents per pound. How much change did she receive from a dollar?

Hidden question: What was the cost of the tomatoes?

Statement No. 1: Let n represent the cost of the tomatoes.

$5 \times .12 = n, \qquad \text{then } n = .60$

Statement No. 2: Let M represent the change she will receive.

$1.00 - .60 = M, \qquad \text{then } M = .40$

The change should be 40 cents.

---

1. Bert and Eddie accepted summer jobs as box boys for the Consolidated Supermarket. They received \$1.12 per hour. The first week Bert worked six hours a day for five days. How much did he earn the first week?
2. During the second week Eddie worked 7 hours a day for five days and also 4 hours on Saturday. What wages did he receive the second week?
3. Each boy receives *overtime pay* for each hour that he works over 40 hours in any week. This overtime pay is \$1.68 per hour. What would Eddie receive for 4 hours of overtime?
4. The third week Bert worked 46 hours. How much did he make that week?
5. One week Eddie worked 48 hours. What were his wages that week?
6. Bert quit his job at the end of eight weeks. He had worked 310 hours at the regular hourly pay and 22 hours at overtime hourly pay. What was Bert's average weekly wage?
7. At the end of 8 weeks Eddie had earned \$375.20. What was his average weekly wage?
8. Eddie wanted to earn a total of \$500 by the end of ten weeks. How much more will he have to earn to reach a total of \$500?
9. Eddie worked 44 hours the 9th week and 48 hours his last week. How much did he earn these two weeks?
10. By how much did he miss his goal of \$500?

11. Joe and Phil were on the football team and were both looking for a summer job. They wanted physical work to keep in condition, so they were glad to get jobs at the Midway Supermarket handling deliveries. They each received $1.20 per hour. The first week Joe worked six hours a day for five days. How much did he earn the first week?

12. The second week, Phil worked 7 hours a day for five days, and 4 hours on Saturday. What wages did he receive the second week?

13. The boys were paid for all overtime (all hours over 40) that they worked each week. If the overtime pay is $1.80 per hour, how much more do the boys receive for each overtime hour as compared with the regular hourly pay?

14. The third week Joe worked 44 hours. He received regular pay for the first 40 hours and overtime pay for the 4 hours of overtime. How much did he earn that week?

15. One week Phil worked 48 hours. What were his wages that week?

16. A gallon of water weighs about 8 pounds. There are 4 quarts to a gallon and 2 pints to a quart. How many pounds will 32 pints of water weigh?

17. When Joe started on an automobile trip he filled the gasoline tank and set the odometer at zero. When he reached Middleton it registered 150 miles. It then took 10 gallons to refill the tank. How many miles per gallon of gasoline was Joe getting ?

18. A machinist is selecting a steel rod that he can cut into 14 pieces each 3 inches long for bolts. How long is the steel rod that he will need to get these 14 pieces? (Actually the rod must be longer because there is some "waste" each time the rod is cut. However, we will not consider this waste.)

19. If you cut 6 pieces each 5 inches long from a brass rod 37 inches long, how long is the piece that is left?

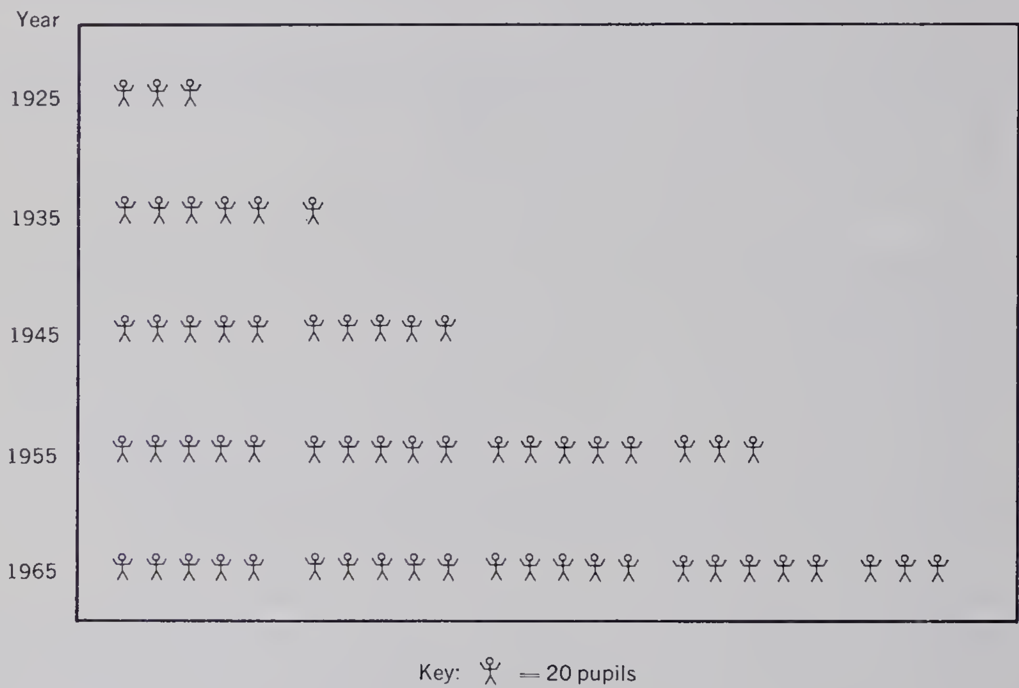
20. A brass rod measures 17 inches. It is to be cut into a number of pieces each measuring 3 inches. How many pieces can be cut from the bar?

21. How long is the piece in Exercise 20 that will be left after the 3-inch pieces have been cut?

22. Tom's car averages 15 miles for each gallon of gasoline. He wants to know how much money he spent for the gasoline he used on a weekend trip. When he started out on Saturday, his odometer read 26,942 miles, and when he returned Sunday, it read 27,167. If gas costs 31¢ per gallon, how much did the gas for the trip cost?

Frequently the data you will use for problem solving comes from a graph. A graph gives us a picture of the relationship between pairs of numbers. For example, the superintendent of the Madisonville schools used a *pictograph* in his report to the school board in order to illustrate the need for a new high school. The pictograph below shows the relationship between certain years and the student enrollment in the old high school. Each little picture (or symbol) in the graph represents 20 pupils. This is called the *key* of the graph.

ENROLLMENT IN THE HANCOCK HIGH SCHOOL



Study the graph and answer these questions about it.

1. The enrollments are given for a period of how many years?

2. How many pupils do the three pictures for the year 1925 represent?

3. Can you tell how many pupils were boys and how many were girls?

4. Before making the graph, the enrollment had been rounded to the nearest 20. How can you tell? Why was this done?

5. How much was the increase in enrollment from 1925 to 1955?

6. How much was the increase from 1925 to 1965?

7. In what ten-year interval was the increase the greatest?

8. Note that the pictures are "grouped" so that there are no more than 5 pictures in a group. List reasons why you think this is done.

9. Explain why the graph is a good way of presenting facts to support the case for a new high school.

A pictograph is useful for presenting data when precise information is not needed. General trends and broad comparisons are readily derived from a pictograph, but unless numbers are small, a pictograph is not useful for presenting detailed information. The steps in making a pictograph are easily followed.

Step a. Select the set of facts you wish to show. Round all numbers to the same place. Generally, round so that the only *non-zero* digits are, at most, the first two digits in the number.

Step b. Arrange a table of the rounded numbers, according to time, size, or other feature.

Step c. Select the key. This key should be chosen so as to represent the greatest number in the table with a reasonable number of symbols and yet show the least number clearly.

Note the key in the graph of Hancock High School enrollment. Would the graph have been as effective if one character had represented 10 pupils? How many characters would be needed for 1965?

Step d. Select a simple symbol or picture to represent each unit.

Step e. The title and key are important parts of the graph. The title should be clearly descriptive of what the graph is intended to show without a detailed explanation.

1. The growth of Hancock High School follows, in general, the national trends in high school enrollment. Since 1900 the national high school population, rounded to the nearest 400,000, has been:

1900 — 800,000	1930 — 4,400,000	1950 — 7,200,000
1910 — 1,200,000	1940 — 7,200,000	1960 — 9,200,000
1920 — 2,400,000		

Using the key:  = 400,000 pupils, construct a pictograph to show the above enrollments. Then study your pictograph and answer these questions about it.

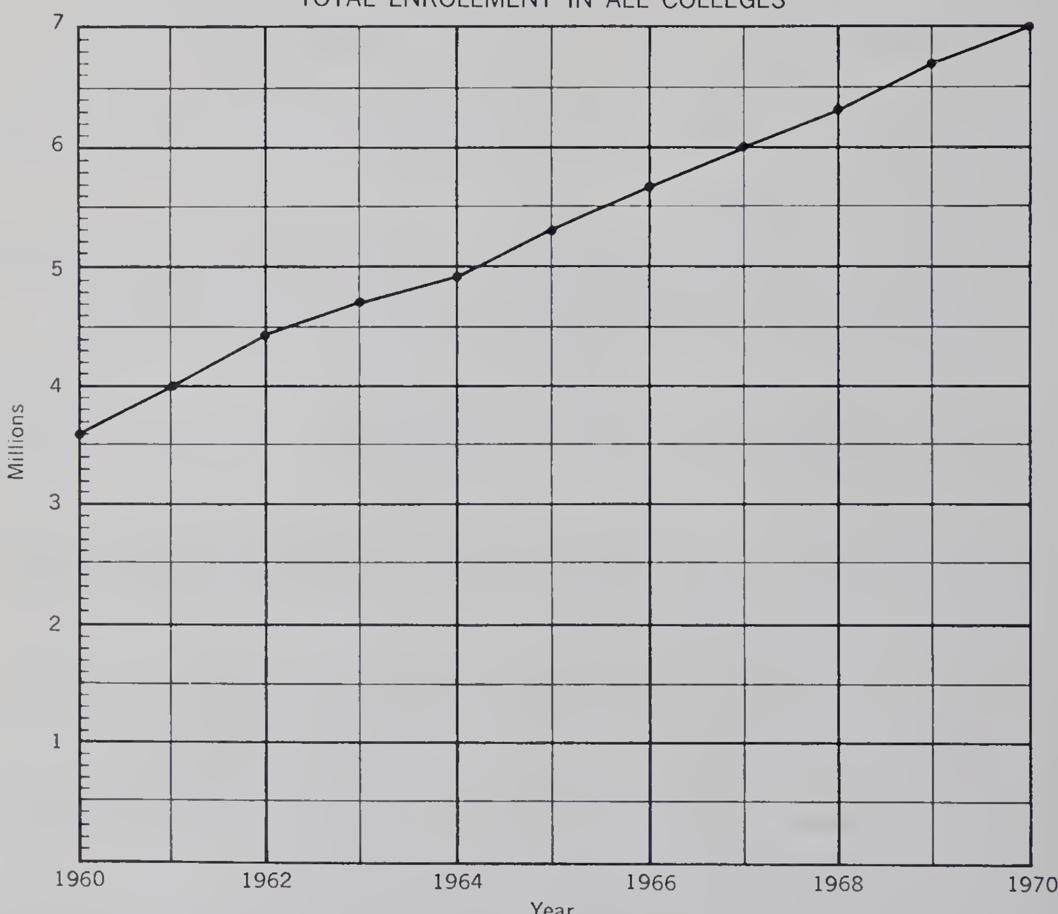
a. How much did enrollments increase in the *decade* (a ten-year period) between 1900 and 1910? between 1950 and 1960?
b. During which decade did enrollments remain about constant? What happened during that decade which might explain this?
c. During which decade did enrollments double?
d. During which decade was the increase greatest?
e. How does the increase from 1920 to 1930 compare to the increase between 1950 and 1960?

A committee in Miss Haynes' class reported on the increase in the number of high school graduates and gave reasons why the number is increasing. They learned that more and more desirable vocations are requiring college preparation, especially for advancement to responsible positions. A national commission that had studied the question predicted that the rise in college enrollments would continue until 1970 at least. The rounded numbers reported and predicted by the commission were:

1960 — 3,600,000	1963 — 4,700,000	1966 — 5,700,000
1961 — 4,000,000	1964 — 4,900,000	1967 — 6,000,000
1962 — 4,400,000	1965 — 5,300,000	1968 — 6,300,000
1969 — 6,700,000	1970 — 7,000,000	

The committee prepared a *line graph* to present these facts and predictions to the class. The graph is shown below. Why do we call this type of graph a line graph?

TOTAL ENROLLMENT IN ALL COLLEGES*



*Enrollments for 1964-1970 are estimated.

The steps in preparing the graph were as follows:

Step a. Graph paper was used because it is "lined." The horizontal scale was labeled on the bottom and the vertical scale on the left side.

Step b. Units of time (1960 through 1970) are on the horizontal scale.

Step c. The number of students was shown on the vertical scale, with divisions selected to include the largest value, that is, 7 million. Each small mark on this scale represents 0.1 million.

Step d. The number of students enrolled each year was located by a dot on the line segment for that year, opposite the number of students indicated on the vertical scale.

Step e. The successive points were joined by segments, making the graph a *broken-line* graph. If the numbers had been continuously changing, the points would have been joined by a smooth curve.

Step f. The horizontal and vertical scales were labeled, and the graph was given a title.

1. The national commission stated that the college enrollments were increasing at the rate of 1 million each three years. Examine the broken-line graph and decide if this is correct.

2. If nothing happens to change conditions, what is your prediction of college enrollments in 1973? What might alter your prediction?

3. During the first month of school John had the scores below on each weekly mathematics quiz. Show John's record on a line graph.

September 5	85
September 12	91
September 19	88
September 26	95

4. The temperature record for one winter day in a midwest city is given below. Show the record on a line graph.

Note: Temperature readings in the table such as $^{-}12$ mean degrees below zero. Other readings in the table that do not have the symbol, –, preceding them represent temperatures above zero.

A.M.

Time	1	2	3	4	5	6	7	8	9	10	11	12
Temp.	$^{-}12$	$^{-}12$	$^{-}11$	$^{-}9$	$^{-}8$	$^{-}7$	$^{-}4$	2	5	8	12	18

P.M.

Time	1	2	3	4	5	6	7	8	9	10	11	12
Temp.	22	20	18	15	11	9	8	6	4	0	$^{-}2$	$^{-}5$

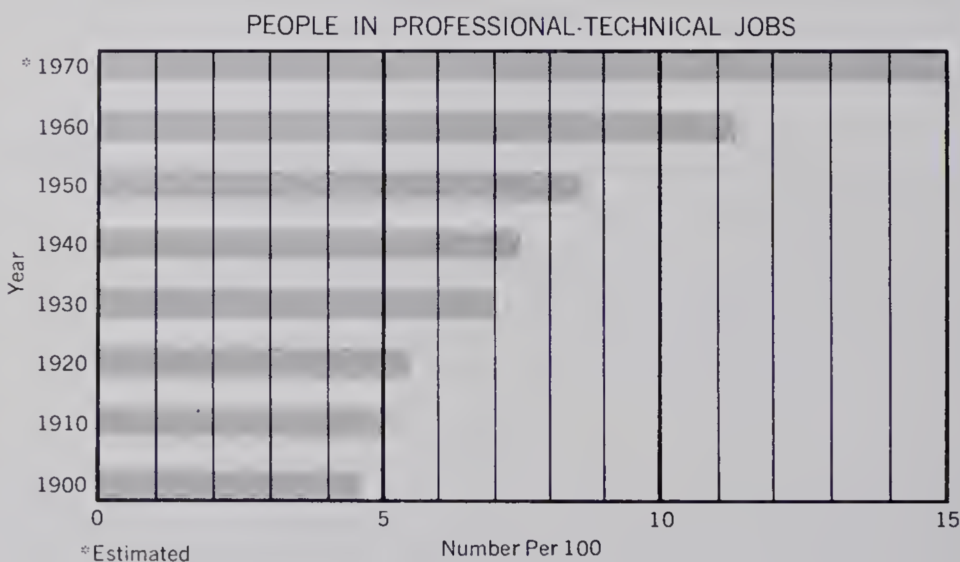
The committee that reported to Miss Haynes' class on college enrollments explained that one reason for the steady increase might be found in the mounting demands for professionally and technically trained workers. This demand has been rising since the beginning of the century, and the rise has been at an increasing rate. The committee found that the number of professional-technical workers per 100 in the workers of this country since 1900 has been (approximately) as follows:

1900 — 4	1920 — 6	1940 — 8	1960 — 11
1910 — 5	1930 — 7	1950 — 9	1970 — 15

(The figure for 1970 was an estimate.)

This means that out of every 100 workers, only 4 held professional-technical jobs in 1900. For every 200 workers in 1900, this would mean that 8 held professional-technical jobs. For every 300 workers in 1900, how many held professional-technical jobs?

The committee presented these facts on a *bar graph* to show what the trend has been (see below). This type of graph was used because it is most effective when general rather than specific comparisons are to be made, or when general trends are to be shown. If the numbers had been large, they would have been rounded, as in constructing a pictograph. In this case, however, the original numbers could be used. Here is the graph as presented by the committee:



These steps were followed in making the graph.

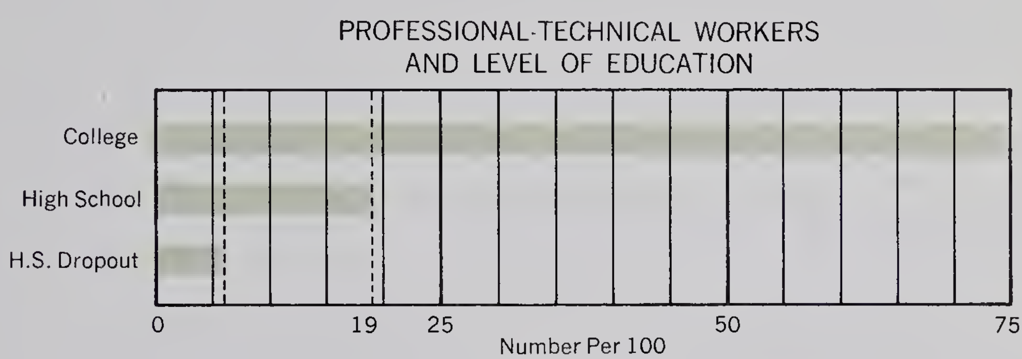
Step a. You will find it convenient to use graph paper. Choose a scale that will show the largest number.

Step b. Label the units along the bottom of the graph if the bars are to be horizontal, or along the left side if they are to be vertical.

Step c. Draw the bars to scale, leaving spaces between the bars at least as wide as the bars.

Step d. Give the graph a title, and label the bars, so that the graph can be understood without further explanation.

1. What do we mean by a professional-technical job? Give some examples.
2. What was the average number of workers per 100 in such jobs in 1910? in 1960? (See the graph on page 66.)
3. What was the increase in number of such workers per 100 between 1900 and 1960?
4. During what ten years was the increase greatest?
5. During what ten years was the increase least?
6. The committee found some facts on the education of workers in professional-technical jobs. They prepared the following graph to show them.



How many high school dropouts were there per 100 workers?

7. The number of workers per 100 with each level of education in other occupations is shown in this table.

Occupation	*Less than High School*	*High School Graduate*	*College*
Proprietors and managers	38	33	29
Clerical and sales	25	53	22
Skilled workers	59	33	8
Services	69	25	6
Semi-skilled	70	26	4
Unskilled	80	17	3

Give an example of an occupation in each category. Prepare a bar graph to show the facts about one of the occupations given above.

Do you remember what a magic square is? Figure 1 is a magic square because you get the same totals when you add the numbers named in each row, column, or diagonal.

16	2	12
6	10	14
8	18	4

Figure 1

Try it and see. It is easy to make a magic square with an odd number of cells. Our procedure in making the magic square is to move diagonally down and to the right to locate each successive numeral. These are the steps for one method of making a magic square, using 1, 2, 3, 4, 5, 6, 7, 8, 9 for a 9-cell square (Figure 2).

Step a. Start with 1 in the middle of the right side. Going down to the right, 2 would fall outside the square. This "fourth" column of the magic square corresponds to the first column of the square. Therefore, place the 2 in the left lower corner cell.

Step b. Moving diagonally down to the right, 3 would fall below the lower center. The "fourth" row of the magic square corresponds to the first row of the square, so 3 goes in the top center cell.

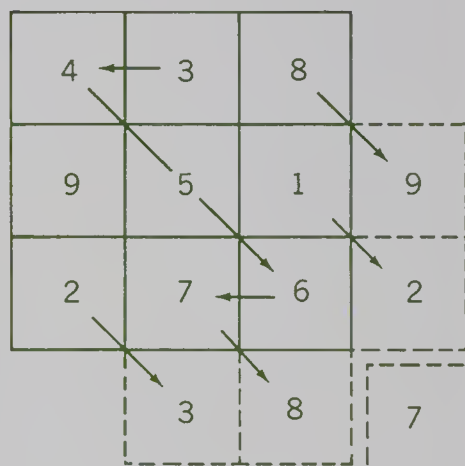


Figure 2

Step c. Moving diagonally down to the right, 4 is blocked by 1. In this situation you go to the left, so 4 is in the upper left corner.

Step d. Moving diagonally down to the right, we have empty cells for 5 and 6.

Step e. Moving diagonally down to the right, we find that 7 falls in a cell which does not correspond to any cell in the magic square. Therefore, proceed as in *Step c* above. That is, place 7 to the left of 6.

Step f. Explain how you locate 8 and 9.

Using the steps above, do the following exercises.

1. Practice making the magic square shown in Figure 2 until you can do it without reading the steps.
2. Now try making a magic square with the odd numbers, beginning with 1 and ending with 17.
3. Try making a magic square with the even numbers, beginning with 10 and ending with 26.

4. An *arithmetic progression* is a sequence of numbers in which the difference between consecutive terms is always the same. 1, 2, 3, 4, 5, etc., and 19, 17, 15, 13, 11, etc., are examples of arithmetic progressions. You can use any arithmetic progression with the proper number of terms to form a magic square. Do the numbers represented in Figures 1 and 2 form an arithmetic progression?

5. If you add the same number to each number in an arithmetic progression you have a new arithmetic progression. Add 3 to each of the natural numbers 1 through 9, and use the progression to make a magic square.

6. If you multiply each number of a magic square by the same number, you have a new magic square. Multiply each term of 1, 3, 5, etc., by 2, and you have 2, 6, 10, etc. Use these to make a magic square.

7. The magic square in Figure 1 was not constructed by the same rule as the one in Figure 2. Study Figure 1 carefully and see how it was constructed. Now select a progression and make a magic square with the same construction.

11	10	4	23	17
18	12	6	5	24
25	19	13	7	1
2	21	20	14	8
9	3	22	16	15

Figure 3

8. In Figure 3 you see a magic square with 25 cells. The construction is in general the same as in Figure 2. Figure out the steps and practice them until you can make one without help.

9. Make a magic square with 25 cells, using the even numbers 2 through 50.

10. In any magic square the sum of the numbers named in each row, column, or diagonal should equal the product of the number named in the center square and the number of cells on one side. Check your magic squares to see if this is true.

Part One

A. *Add:*

1.	2.	3.	4.	5.	6.
57	326	$7.65	393	$ .12	$3.89
53	500	7.09	202	8.05	.16
85	747	2.75	158	9.69	.75
52	653	9.87	37	.58	9.54
68	938	5.39	205	2.00	.79
73	391	9.83	770	.11	.67
19	717	6.72	19	7.20	.09

B. *Subtract:*

1.	3.	5.	7.
5725	465,258	$1409.25	$1750.00
2113	78,249	1209.27	893.05

2.	4.	6.	8.
$400.00	23,841	$342.65	754,327
2.45	22,659	68.38	403,265

C. *Multiply:*

1.	3.	5.	7.
3529	9343	$88.06	$93.56
52	96	716	6108

2.	4.	6.	8.
$43.80	9761	$56.20	5214
270	58	602	36

Find the products with as little written work as possible.

9. 300 × 103
10. 200 × 315
11. 4002 × 16
12. 103 × 25
13. 300 × $2.10
14. 2000 × 13

D. *Divide:*

1. $32\overline{)1472}$
2. $218\overline{)\$6522.56}$
3. $293\overline{)\$404.34}$
4. $784\overline{)\$8796.48}$
5. $428\overline{)\$3830.60}$
6. $963\overline{)\$2359.35}$

Part Two

A. List the numerals 1 through 4 on a sheet of paper. After each numeral place a letter to indicate which of the expressions at the right is defined by the expression having that numeral.

1. The statement of a mathematical relationship

2. A letter that holds the place for a number

3. A statement that is neither true nor false until certain requirements are met

4. A statement that two expressions name the same number

a. equation
b. mathematical problem
c. mathematical statement
d. open sentence
e. false statement
f. true statement
g. variable

B. Given the relationships

$$\text{addend} + \text{addend} = \text{sum}$$
$$\text{and}$$
$$\text{factor} \times \text{factor} = \text{product}$$

state which term is missing in each of these conditional statements. Then find the value that makes the statement true.

1. $19 + 13 = n$
2. $23a = 69$
3. $48 \div 6 = x$
4. $35 - n = 21$
5. $y \div 45 = 16$
6. $26 - 13 = x$
7. $343 \div a = 7$
8. $240 \div n = 15$
9. $y \div 18 = 9$
10. $n - 15 = 24$
11. $18x = 162$
12. $15 \times 7 = b$
13. $261 - n = 85$
14. $56 \div y = 7$
15. $x - 25 = 47$
16. $17n = 136$

C. Graphing

The population of the United States, according to the census bureau, has been as follows during the present century:

1900 — 75,994,575
1910 — 91,972,266
1920 — 105,710,620
1930 — 122,775,046
1940 — 131,669,275
1950 — 150,697,361
1960 — 179,323,175

Draw a line graph to show these facts. (Round the numbers to the nearest million.)

Part Three

1. A total of 64 boys turned out for football at Washington High School. It costs $90 to equip each one. About how much will it cost to outfit the squad?
2. Mr. Jensen is a carpenter who has been asked to estimate the cost of a job that will take him 26 days to complete. He figures on earning $28.75 a day. How much should he estimate for labor on the job?
3. Mr. Henderson has 60 acres planted in corn. He expects the crop to average 65 bushels an acre. If it does, and corn sells for $1.37 a bushel, how much will his crop be worth?
4. Jim Anderson wants a job to earn Christmas money. He is offered a chance to work 16 hours a week during the afternoons and on Saturdays at a gas station, at $1.85 an hour. There are 8 weeks remaining until Christmas. How much can he earn if he takes the job?
5. Jane Andrews has been saving the money she earned from babysitting since July. She has worked 120 hours at $1.25 per hour. How much has she saved?
6. On a camping trip Harry hiked 15 miles in 5 hours. At the same rate how far could he hike in 8 hours?
7. An airplane made a trip of 1400 miles in 4 hours. How many miles did it travel per hour?
8. Helen worked 14 hours last week at $1.40 an hour. How much of her earnings were left after she had paid $9.98 for a pair of shoes?
9. Mr. Allen sold 11 cows for $325 each. He used part of the money to pay $2250 for a tractor. How much was left?
10. A truck driver drove from Saint Louis to Minneapolis, a distance of 600 miles, in 15 hours. At the same rate, how long would it take him to drive from Minneapolis to Denver, a distance of 1000 miles?

CHAPTER THREE

MULTIPLICATION AND DIVISION

WORDS TO WATCH FOR

additive identity	*division*	*multiplicative identity*
associative property	*divisor*	*product*
commutative property	*identity element*	*quotient*
distributive property	*inverse*	*remainder*
dividend	*multiplication*	

Multiplication and division are operations that you have been using for a long time. However, there are some interesting and important principles and properties of these operations that you may not have become aware of. By learning something about them you will be able to use the operations more effectively.

For example, multiplication may be thought of as a series of repeated additions using the same addend.

PROBLEM

Edward was helping unload boxes of canned goods from the trucks at the store where he works part time. To keep count, he piled only 9 boxes in a stack. When the truck was unloaded he had 6 stacks. To find the number of boxes he had a choice: either add, or multiply. This fact is illustrated by the Examples at the right.

1. Explain each operation. Which looks easier? Why?

2. Show how Example 1 and Example 2 would look if there had been 8 stacks of 7 boxes each.

EXAMPLES

1.
$$\begin{array}{r} 9 \\ 9 \\ 9 \\ 9 \\ 9 \\ 9 \\ \hline 54 \end{array}$$

2.
$$\begin{array}{r} 9 \\ \times\ 6 \\ \hline 54 \end{array}$$

3. In the previous chapter you studied the relationship,

"factor $\times$ factor = product" or $x \times y = p$

In Exercise 2,

x = the number of small sets of boxes (stacks)
y = the number of elements (boxes) in each set
p = the number of elements (boxes) when the small sets are combined into a larger set

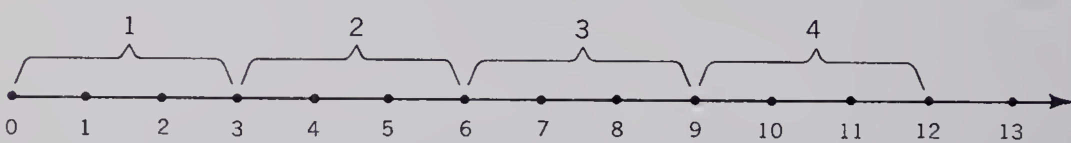
State the number associated with each: x, y, and p.

4. In the assembly hall at Washington High School there are 17 rows of seats with 25 seats in each row. When all the seats are filled, how many people are in the room?

5. In Exercise 4, you could either add or multiply to find the answer. Which is easier?

6. In Exercise 4, what number is associated with each: x, y, and p? (As you will see shortly, it does not matter which of the factors you associate with x or y.)

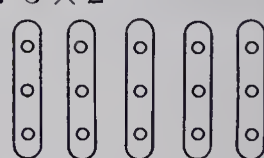
7. Multiplication can readily be represented on the number line. Here you have a representation of the operation: $4 \times 3 = 12$.



You can see that the result is the same as if you had counted by 4's, and made 3 counts. If x is 3, what is y? What is p?

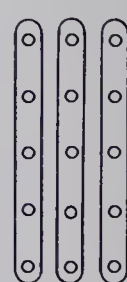
8. Use the number line to show each of the following operations:

a. 2×4 **b.** 5×2 **c.** 2×3 **d.** 3×2



9. If we are combining 5 sets of 3 members each, and x is the number of sets, what numbers are associated with x, y, and p?

10. Suppose that instead of 5 sets of 3 members each, we rearrange the Figure to have 3 sets of 5 members each. If x is the number of sets, what numbers are associated with x, y, and p?



11. Is the result the same if we combine 5 sets of 3 members each, as if we combine 3 sets of 5 members each? Represent with a figure, 6 sets of 4 members, and show that it can be rearranged to represent 4 sets of 6 members each. What is the number of members in the set when the smaller sets are combined?

12. Exercises 9, 10, and 11 illustrate the *commutative property of multiplication:*

The order of arranging the factors does not affect the product. In symbols (x and y represent any whole number):

$$x \times y = y \times x$$

Does $9 \times 7 = 7 \times 9$? Illustrate this property with other factors.

13. You have used the commutative property of multiplication in simplifying multiplication exercises. Select the simpler form and carry out this multiplication:

$$\begin{array}{r} 324 \\ \times\ 9 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 9 \\ \times\ 324 \\ \hline \end{array}$$

14. In one classroom there are 7 rows of chairs with 5 chairs in a row. In another classroom there are 5 rows of chairs with 7 chairs in a row. How many chairs are there in each classroom?

15. Since the factors may be taken in any order, we only identify them as x and y when there is a reason for keeping them distinct. Make use of the commutative property to set up and perform each of the following operations in the simplest way:

a. 674×7 **b.** 12×891 **c.** 259×7 **d.** 137×100

16. Since multiplication is a short-cut for addition, you would expect that the commutative property should also hold for addition. Does $8 + 5 = 5 + 8$?

17. Use the number line to show that $5 + 2 = 2 + 5$.

18. The *commutative property of addition* may be stated as follows:

The order in which addends are arranged does not affect the sum. In symbols (a and b represent any whole number):

$$a + b = b + a$$

Substitute the given values for a and b, and show that the commutative property holds for each of the following:

a. $a = 11, b = 5$ **c.** $a = 15, b = 3$
b. $a = 7, b = 9$ **d.** $a = 13, b = 7$

19. Jim had \$8 and earned \$5 more. Henry had \$5 and earned \$8. Set up a mathematical sentence to show how much each boy had.

You know that multiplication is used to combine sets each having the same number of elements. Read each of the following problems and tell whether you should multiply or add to find the answer, and explain why. Then find the answer.

1. Mary's lunches last week cost: Monday, 27 cents; Tuesday, 35 cents; Wednesday, 24 cents; Thursday, 32 cents; Friday, 30 cents. How much did she spend for lunches last week?

2. George works at a filling station for two hours after school each day, five days a week. He is paid 85 cents an hour. How much does he earn in a week?

3. The Adams Elementary School has six grades. There are 34 pupils in Grade 1; 36 in Grade 2; 33 in Grade 3; 38 in Grade 4; 36 in Grade 5; and 35 in Grade 6. What is the total number of pupils in the school?

4. Last week the basketball team in Carl's room took a physical examination. The weight in pounds of the boys on the first team was as follows: Carl, 95; Henry, 92; George, 108; Bob, 97; and Jack, 110. What was the total weight of the first team?

5. The trees in an orchard are arranged in rows of 26 trees each. There are 18 rows. How many trees are there in the orchard?

6. Henry sells papers every afternoon. Last week his earnings were: Monday, $1.15; Tuesday, 95 cents; Wednesday, $1.20; Thursday, $1.15; Friday, 85 cents. How much did he earn during the week?

7. A grocery store purchased 10 gallons of milk at 55 cents per gallon and sold it at 26 cents per quart. What was the cost of the milk? What was the selling price of the milk?

8. A boys' club on a pack trip hiked into the mountains four days and averaged 7 miles per day. How far did they hike?

9. On another 4-day trip the boys traveled 11 miles the first day, 10 miles the second day, 7 miles the third day, and 14 miles the last day. How far did they hike?

10. At the junior high math tournament each school entered 6 pupils, and 12 schools entered the tournament. How many pupils were in the tournament?

11. The halfback on the football team carried the ball 8 times and gained and lost yardage as follows: gained: 6, 11, 3, 8, 23; lost: 2, 3, and 7. How many yards did he gain? How many yards did he lose?

12. Make up a general rule to help you decide when you should add and when you should multiply.

How well can you use the steps for solving applied problems?

STEPS FOR SOLVING APPLIED PROBLEMS

1. Understand the problem.
2. Note what the problem asks for.
3. Look for hidden questions.
4. Estimate a reasonable answer.
5. Set up and solve the conditional statement(s).
6. Check your answer.

1. Mary worked 14 hours a week last year as a secretary in a doctor's office. She was paid $1.35 an hour. How much did she earn per week?

2. Jim works at a warehouse loading trucks 3 hours a day after school for 5 days a week, and 4 hours on Saturday. How many hours does he work each week?

3. Jim is paid $1.30 an hour. How much does he earn on each school day? How much does Jim earn each week?

4. Last year the school term lasted 40 weeks. Jim worked at the warehouse each week. How much did he earn during the school term?

5. George wants to buy a set of golf clubs for $47.70 and a golf bag for $9.90. To earn the money, he is working at 90 cents an hour in the stockroom at the supermarket. How many hours must he work to pay for the golf clubs and bag?

6. George works in the stockroom 2 hours each day after school and 6 hours on Saturday. How many weeks will it take him to earn the money for his golf clubs and bag?

7. Mildred can type 50 words a minute. She was employed to type a manuscript containing 12,000 words. How many hours did it take her to do the typing?

8. Mildred was paid $4.80. How much did she earn per hour?

9. Joe worked as a machinist's helper last summer at $1.10 per hour. For all the time over 40 hours per week he received $1.65 per hour (overtime pay). What did Joe receive for 3 hours of overtime?

10. One week Joe worked 46 hours. What did he earn for the week?

11. After two months Joe received a raise to $1.20 per hour, and a 5-cent raise for overtime. How much would he earn if he put in a 47-hour week?

The fifth of the problem-solving steps is to set up and solve the conditional statement or statements. The conditional statement has a special value in problem solving because it clearly reveals the mathematical relationships in the problem situation and removes the confusing elements in the problem that are not related to the solution. Each hidden question calls for a conditional statement. Be sure to set up the statements for each of these problems.

EXAMPLE

Mr. Adams lives in a state where the tax on gasoline is $.11 per gallon. His car can travel 15 miles on a gallon of gasoline. On a trip of 1200 miles, how much tax will Mr. Adams pay on the gasoline he uses?

Hidden question: How many gallons of gasoline did he use?

Let n represent the number of gallons he used.

$$1200 \div 15 = n, \qquad n = 80$$

Question: How much is the tax on 80 gallons?

Let N represent the tax (in cents) on 80 gallons.

$$80 \times 11 = N, \qquad N = 880$$

The tax is $8.80.

Be sure to set up the conditional statement or statements in solving each of the following problems.

1. Mr. Johnson pays a rent of $115 per month on his house. This is how much rent per year?

2. Henry takes care of lawns afternoons and Saturdays at $1.10 an hour. Last week he worked 12 hours. How much did he earn?

3. At the same rate, how many hours will it take Henry to earn $33?

4. Mr. Erickson earns $125 a week. There are 52 weeks in a year. How much does Mr. Erickson earn per year?

5. Jim's car travels 16 miles on a gallon of gasoline. How far can he travel on 12 gallons?

6. Margaret works part time as a check-out clerk in a supermarket. Last month she worked 15 hours during the first week, 22 hours the second, 18 hours the third, and 15 hours the fourth. She is paid 90¢ per hour. How much did she earn last month?

7. Mike washes and polishes cars on Saturdays, charging $2.25 per car. He plans to purchase a bicycle costing $45. How many cars must he wash and polish to pay for the bicycle?

8. It takes Mike 2 hours to wash and polish a car. How many hours will it take him to earn $45?

9. Jane wants to buy a camera that will cost $25. She earns $1.25 an hour baby-sitting, and can baby-sit 4 hours per week. How many weeks will it take her to earn enough to buy the camera?

10. Arthur sells papers after school five days per week. He earns 2¢ for each paper that he sells. The number of papers he sold each day last week were: Monday, 44; Tuesday, 56; Wednesday, 38; Thursday, 50; Friday, 68. What were his total earnings for the week?

11. Tim was offered a job at $1.15 an hour. If he can work 10 hours a week for 8 weeks, how much can he earn?

12. Janet Blake has been saving money she earned doing typing jobs. She worked 120 hours at $1.05 an hour. How much has she earned?

13. Mr. Henderson is a carpenter and his average earnings are $20 a day. How much should he ask for doing a job that he estimates will take him 26 days?

14. The Wilson High School student organization sold 800 season football tickets at $1.50 each. What was the total collected?

15. A car traveling 45 miles an hour uses a gallon of gasoline each 15 miles. At that rate, how much gasoline will it use in 3 hours?

16. A one-pound can of coffee is marked 90¢. A two-pound can is marked $1.75. How much is saved by purchasing a two-pound can rather than two one-pound cans?

17. Mike plans to buy a bicycle that will cost him $36. He now has $12. He has a job working after school 2 hours a day, Monday through Friday, at 80¢ an hour. How many weeks will it take him to earn the remainder of the cost of the bicycle?

18. Mr. Jones, the grocer, buys sugar at the rate of $5.35 per one hundred pounds. What does he pay for 1900 pounds?

19. If the price of beef cattle is $17 per 100 pounds, what is the value of a steer weighing 1300 pounds?

20. What is the value of a carload of hogs weighing 21,300 pounds when the price is $15 per hundred pounds?

21. An automotive store, during their tire sale, priced black sidewall tires at $15.88 each. How much will 4 tires cost?

22. Henry prefers white sidewall tires. These are on sale for $16.33. How much more will 4 white sidewall tires cost than 4 black sidewall tires?

Suppose you are asked to perform this exercise:

$$8 \times 5 \times 7 = n$$

You can multiply only two factors at a time. Does it matter which two you start with? We use parentheses to indicate which we multiply first:

$$\begin{aligned} (8 \times 5) \times 7 &= n \\ 40 \quad \times 7 &= n \\ 280 &= n \end{aligned} \qquad \begin{aligned} 8 \times (5 \times 7) &= n \\ 8 \times \quad 35 \quad &= n \\ 280 &= n \end{aligned}$$

The above illustrates the *associative property of multiplication:*

We may group the factors in any order without affecting the product. In symbols (a, b, and c represent any whole number):

$$(a \times b) \times c = a \times (b \times c)$$

EXAMPLE

$$7 \times 4 \times 25 = n$$

If we first take 7×4 we have: $28 \times 25 = n$
You would probably have to use pencil and paper to compute the result. However, if we group as follows, the solution is simpler:

$$\begin{aligned} 7 \times (4 \times 25) &= n \\ 7 \times \quad 100 \quad &= n \\ 700 &= n \end{aligned}$$

Sometimes you can profitably use both the commutative and associative properties to change the order of grouping and simplify an operation.

EXAMPLE

$5 \times 13 \times 20 = n$	
$13 \times 5 \times 20 = n$	Commutative property
$13 \times (5 \times 20) = n$	Associative property
$13 \times \quad 100 \quad = n$	
$1300 = n$	

Use the associative and commutative properties to rearrange and regroup in order to simplify each multiplication.

1. $2 \times 8 \times 50 = n$

2. $20 \times 13 \times 5 = n$

3. $5 \times 17 \times 40 = n$

4. $75 \times 11 \times 4 = n$

5. $50 \times 9 \times 2 = n$

6. $4 \times 15 \times 25 = n$

The associative property also holds for addition:

We may group the addends in any order without affecting the sum. In symbols (a, b, and c represent any whole number):

$$(a + b) + c = a + (b + c)$$

EXAMPLES

1.

$$\begin{aligned} 8 + 6 + 4 &= n \\ 8 + (6 + 4) &= n \\ 8 + \quad 10 \quad &= n \\ 18 &= n \end{aligned}$$

Or, with another grouping:

$$\begin{aligned} (8 + 6) + 4 &= n \\ 14 \quad + 4 &= n \\ 18 &= n \end{aligned}$$

We can also combine the commutative and associative properties to simplify addition:

2.

$16 + 27 + 4 = n$	
$16 + 4 + 27 = n$	Commutative property
$(16 + 4) + 27 = n$	Associative Property
$20 \quad + 27 = n$	
$47 = n$	

Use the associative and commutative properties to rearrange and regroup in order to simplify each addition.

7. $17 + 16 + 13 = n$ **10.** $14 + 28 + 6 = n$ **13.** $15 + 18 + 5 = n$
8. $25 + 26 + 5 = n$ **11.** $19 + 11 + 15 = n$ **14.** $26 + 37 + 4 = n$
9. $35 + 19 + 15 = n$ **12.** $4 + 27 + 46 = n$ **15.** $12 + 14 + 18 = n$

Sometimes multiplication can be simplified if one of the factors is renamed as two factors.

EXAMPLE

$24 \times 75 = n$	
$(6 \times 4) \times 75 = n$	Renaming 24 as 6×4
$6 \times (4 \times 75) = n$	Associative property
$6 \times \quad 300 \quad = n$	
$1800 = n$	

Rename one of the factors and find the products.

16. 24×25 **19.** 16×250 **22.** 44×25 **25.** 48×25
17. 26×50 **20.** 28×500 **23.** 38×50 **26.** 36×75
18. 60×25 **21.** 32×50 **24.** 66×500 **27.** 16×25

Many people like to "work for themselves" rather than for wages or salaries. Jobs that are taken on for a stated price are said to be *contracted*. Building or making extensive repairs on a home may be done by a contractor. The government gives out contracts to build roads and buildings. The contractor employs workers, like carpenters, plumbers, and so on, to do part or all of the work. He calculates how much it will cost him to do the job, adds a reasonable profit, and submits his estimate as a *bid*. If the bid is accepted he enters into a *contract* to do the job. Work that is paid for according to the amount done is called *piece work*.

1. Phil agreed to take care of Mr. Henderson's lawn for the three summer months for $36.00. Although there was no written paper, Phil *contracted* to take the job. How much did he charge per month?

2. Phil kept track of the time he worked on the Henderson's lawn to see if his contract was a profitable one. He worked 13 hours in June, 20 hours in July, and 17 hours in August. How much did he earn per hour?

3. Henry agreed to paint the garage for $40. He bought 2 gallons of paint costing $4.80 per gallon. It took him 20 hours to do the job. How much did he earn per hour?

4. Eric took a job picking apricots for his uncle. He was paid 30 cents per box, and could pick 4 boxes per hour. How much did he earn in an 8-hour day?

5. Henrietta types letters for her father at 40 cents per page. She can type an average of 5 pages per hour. How many hours must she work to earn $30.00?

6. Alice has a job addressing envelopes for $10.00 per thousand envelopes. She can address 125 per hour. How much does she earn per hour?

7. Mr. Dale works in a machine shop. He is paid $3.50 for every one hundred pieces he turns out. In four days he turned out 4100 pieces. How much did he earn?

8. Mr. Warner assembles instruments in an electric shop. He is paid 40¢ per instrument. Normally, he can assemble 6 instruments in an hour. At this rate, how much can he earn in a 40-hour week?

9. Mary Jensen picked string beans last summer at 75¢ a basket. In one week she had the following record:

Monday	18 baskets	Thursday	17 baskets
Tuesday	15 baskets	Friday	16 baskets
Wednesday	14 baskets	Saturday	8 baskets

How much did she earn each day?

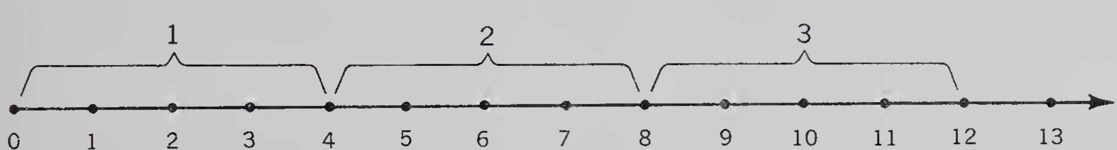
In the previous chapter you found that, using the "if-then" type of reasoning, you could always set up two statements that were equivalent to a multiplication statement.

$$\begin{array}{lll} \text{If } 18 \times 17 = 162 & \qquad & \text{If } x \times y = p \\ \text{then } 18 = 162 \div 17 & & \text{then } x = p \div y \\ \text{and } 17 = 162 \div 18 & & \text{and } y = p \div x \end{array}$$

The second and third statements in each column are statements about division. They tell you that if the product and one factor are known, you can divide the product by the known factor and obtain the unknown factor.

Because division is used in this way to "undo" the operation of multiplication, it is called the *inverse operation* of multiplication. It is usually convenient to use the terms product and factor, as in the second and third statements above, even though the operation is division. On the other hand it is important to understand the terminology of division: the product is the *dividend;* the known factor is the *divisor;* and the unknown factor is the *quotient.*

In a division situation we are finding how many smaller groups are in a larger group. For example, how many groups of 4 are in a group of 12? Referring to the number line you can see at once there are 3. Use the number line to show how many groups of 3 there are in a group of 12.



1. Use the number line to illustrate each of the following:

a. $10 \div 5$ **b.** $15 \div 3$ **c.** $12 \div 3$ **d.** $16 \div 4$

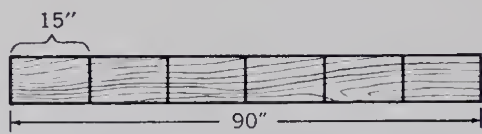
2. The Math Club at Gage High School is going to a science exhibit at the museum. There are 20 members in the club, and 4 can go in one car. How many cars are needed?

3. Check your answer to Exercise 2 by making a sketch of 20 x's with 4 x's in each row. If x is the number of persons in a car, what is y? What is p?

Multiplication may be thought of as repeated addition. Since division is the inverse operation of multiplication, we should expect that division is repeated subtraction. Let us see if this is true.

Raymond is putting bookshelves in his study. Each one is 15″ long. He is cutting them from a board that is 90″ long. How many shelves will he be able to cut from this board?

One way to figure this out would be to simply mark off 15″ lengths on the 90″ board, as follows:



Another way to find out, would be to calculate how long the board would be after each 15″ piece is cut off. This is shown in Example 1. You can see that he would get six shelves. A quicker way is to divide: $90 \div 15 = 6$. This is shown in Example 2. By comparing these two Examples you can see that division is the quick way to carry out repeated subtraction.

EXAMPLES

1.

Length	90″	
First Cut	15″	(1)
Remaining	75″	
Second Cut	15″	(2)
Remaining	60″	
Third Cut	15″	(3)
Remaining	45″	
Fourth Cut	15″	(4)
Remaining	30″	
Fifth Cut	15″	(5)
Remaining	15″	(6)

2.

$$\begin{array}{r} 6 \\ 15\overline{)90} \\ \underline{90} \end{array}$$

In Exercises 1–10, see how many times you can subtract the second number named from the first number named in each pair. Then divide the first number by the second. Are the results the same?

1. 45; 9

2. 72; 12

3. 52; 13

4. 51; 17

5. 54; 18

6. 92; 23

7. 78; 26

8. 150; 30

9. 72; 36

10. 57; 57

Find the answers to these problems by using repeated subtraction.

11. Mr. Jones has a load of 288 pounds of apples from his orchard. He is putting the apples in sacks of 48 pounds each. How many full sacks will he have?

12. How many pieces 8″ long can be cut from a ribbon 104″ long?

13. Suppose you wanted to find out how many 7's there are in 84 by repeated subtraction. The Example at the right shows how you can first take ten 7's or 70, and then two 7's or 14. What is the quotient?

EXAMPLE

$$\begin{array}{rr} & 84 \\ 10 \times 7 = & \underline{70} \\ & 14 \\ 2 \times 7 = & 14 \end{array}$$

Use the method of repeated subtraction in Exercises 14–16.

14. At $6.00 per day, how long will it take to earn $96?

15. What is $126 \div 9$?

16. Divide 104 by 8.

17. Example 1 shows $5616 \div 24$, using continued subtraction. In Example 2, this is shortened to the long division process that you are familiar with. In Example 1, how would you determine how many hundreds to multiply by 24? how many tens? how many ones?

EXAMPLES

1.

$$\begin{array}{rr} & 5616 \\ 200 \times 24 = & \underline{4800} \\ & 816 \\ 30 \times 24 = & \underline{720} \\ & 96 \\ 4 \times 24 = & \underline{96} \end{array}$$

2.

$$\begin{array}{r} 234 \\ 24\overline{)5616} \\ \underline{48} \\ 81 \\ \underline{72} \\ 96 \\ \underline{96} \end{array}$$

18. In Example 2 above, why is it not necessary to use zeros to indicate hundreds and tens?

19. Find the quotients, using the process of long division.

a. $1694 \div 14$

b. $3264 \div 16$

c. $901 \div 17$

d. $666 \div 37$

e. $1935 \div 45$

f. $2808 \div 24$

g. $414 \div 23$

h. $1197 \div 57$

i. $2268 \div 63$

j. $3311 \div 77$

1. In most of the division exercises you have been working, the dividend was exactly divisible by the divisor (the remainder was zero). Of course, this is not always the case. What is the remainder in the Example at the right? What is the quotient? One way to write the result of the division is:

$$496 \div 23 = 21 \text{ r } 13$$

EXAMPLE

$$\begin{array}{r} 21 \\ 23\overline{)496} \\ \underline{46} \\ 36 \\ \underline{23} \\ 13 \end{array}$$

Use this form in the following exercises where there is a remainder.

Divide:

2. $686 \div 14$

3. $1275 \div 25$

4. $1296 \div 36$

5. $1872 \div 18$

6. $4833 \div 27$

7. $5688 \div 72$

8. $958 \div 29$

9. $1531 \div 23$

10. $1642 \div 31$

11. $2071 \div 19$

12. $1648 \div 41$

13. $5676 \div 43$

14. $3682 \div 34$

15. $5719 \div 48$

16. $6320 \div 51$

17. $7617 \div 65$

QUESTIONS FOR EXPLORATION

Do the commutative and associative properties hold for division? In testing to see if a property holds, one example showing that it does *not* hold is sufficient for final proof.

1. Does the commutative property hold for division?
If so, then: $12 \div 3 = 3 \div 12$
Is this a true statement? Complete the computation and see.

2. Test with other numbers to show that the commutative property does not hold for division.

3. Does the associative property hold for division?
If so, then: $(24 \div 6) \div 2 = 24 \div (6 \div 2)$
That is: $4 \div 2 = 24 \div 3$
Is this a true statement? Complete the computation and see.

4. Test with other numbers to show that the associative property does not hold for division.

The *distributive* properties of multiplication with respect to addition and division with respect to addition are properties that you have used continually. Consider the following: $3 \times 32 = n$ We make use of the distributive property by actually performing it as: $3 \times (2 + 30) = n$

$$\begin{array}{rl} 32 & \\ \times\ 3 & \\ \hline 6 & = 3 \times 2 \\ 90 & = 3 \times 30 \\ \hline 96 & \end{array}$$

While you do not ordinarily set down the operation in this form, you will recognize that this is the way in which the computation is actually carried out. In other words,

$$\begin{aligned} 3 \times 32 &= 3 \times (30 + 2) \\ &= (3 \times 30) + (3 \times 2) \end{aligned}$$

Stated as a property:

Multiplication is distributive with respect to addition. In symbols (a, b, and c represent any whole number):

$$a \times (b + c) = (a \times b) + (a \times c)$$

or $$a \times (b + c) = ab + ac$$

The following is also true because of what property?

$$(b + c) \times a = (b \times a) + (c \times a)$$

This property is frequently useful in simplifying a multiplication problem.

EXAMPLES

1. Multiply: $6 \times 64 = ?$

$$\begin{aligned} 6 \times 64 &= 6 \times (60 + 4) \\ &= (6 \times 60) + (6 \times 4) \\ &= \quad 360 \quad + \quad 24 \\ &= 384 \end{aligned}$$

2. Multiply: $259 \times 4 = ?$

$$\begin{aligned} 259 \times 4 &= (250 + 9) \times 4 \\ &= (250 \times 4) + (9 \times 4) \\ &= \quad 1000 \quad + \quad 36 \\ &= 1036 \end{aligned}$$

1. Evaluate each of the following by first performing the operation within the parentheses.

a. $8 \times (10 + 5)$	**d.** $17 \times (23 + 7)$	**g.** $8 \times (19 + 1)$
b. $(6 + 8) \times 3$	**e.** $(30 + 3) \times 12$	**h.** $(4 + 60) \times 5$
c. $24 \times (2 + 1)$	**f.** $3 \times (15 + 8)$	**i.** $(100 + 200) \times 10$

2. Evaluate each of the above by using the distributive property of multiplication with respect to addition. Do your answers agree with the answers for Exercise 1?

3. Use the distributive property to find the product in each of the following. Refer to the Examples on page 87.

a. 4×35	**e.** 47×5	**i.** 8×35
b 96×8	**f.** 6×85	**j.** 3×126
c. 7×56	**g.** 92×7	**k.** 28×6
d. 3×59	**h.** 4×132	**l.** 9×52

Another Distributive Property

Is division distributive with respect to addition? That is, does $154 \div 7 = (140 \div 7) + (14 \div 7)$?

Stated as a property:

Division is distributive with respect to addition. In symbols (a, b, and c represent any whole number, $c \neq 0$):

$$(a + b) \div c = (a \div c) + (b \div c)$$

Note: Division by zero is not defined, so it is necessary to state that $c \neq 0$ in the above. We will discuss zero and division later in this chapter.

You use the distributive property whenever you perform division using any other than single-digit numerals.

EXAMPLE

Divide: $175 \div 5$

175 is renamed as: $(150 + 25)$

150 is the greatest multiple of (5×10) that can be taken from 175. The remainder is 25.

$$\begin{aligned}(150 + 25) \div 5 &= (150 \div 5) + (25 \div 5)\\ &= \quad 30 \quad + \quad 5\\ &= 35\end{aligned}$$

Carry out the division of 175 by 5, and explain why it was renamed as explained above. With larger numbers, as you might expect, the dividend would be renamed as the sum of 3 or more numbers.

EXAMPLE

Divide: $798 \div 6$

$$798 \text{ is renamed as } (600 + 180 + 18)$$

These are respectively: the greatest multiple of (6×100) that can be taken from 798; the greatest multiple of (6×10) that can be taken from $798 - 600 = 198$; and the remainder, 18.

$$(600 + 180 + 18) \div 6 = (600 \div 6) + (180 \div 6) + (18 \div 6)$$

Complete the division.
Carry out the division by long division, and explain why 798 was renamed as explained above.

---

4. Evaluate each of the following by first performing the operation within the parentheses.

a. $(3 + 12) \div 3$
b. $(8 + 16) \div 4$
c. $(15 + 25) \div 5$
d. $(32 + 54) \div 2$
e. $(26 + 52) \div 13$
f. $(100 + 75) \div 25$
g. $(36 + 78) \div 6$
h. $(21 + 35) \div 7$
i. $(45 + 72) \div 9$

5. Evaluate each of the above by using the distributive property of division with respect to addition. Do your answers agree with the answers for Exercise 4?

6. Use the distributive property to find the following quotients. Refer to the Examples on pages 88 and 89.

a. $192 \div 6$
b. $117 \div 9$
c. $266 \div 7$
d. $608 \div 8$
e. $434 \div 7$
f. $462 \div 6$
g. $288 \div 8$
h. $396 \div 11$
i. $904 \div 8$
j. $762 \div 6$
k. $790 \div 5$
l. $1239 \div 7$

7. Evaluate each pair below by first performing the operation within the parentheses.

a. $8 \times (4 - 3)$; $(8 \times 4) - (8 \times 3)$
b. $12 \times (5 - 2)$; $(12 \times 5) - (12 \times 2)$
c. $(20 - 4) \times 14$; $(20 \times 14) - (4 \times 14)$
d. $(36 - 15) \times 2$; $(36 \times 2) - (15 \times 2)$

Does multiplication appear to be distributive with respect to subtraction?

Multiplication and Division

A. Find the products.

1. $\begin{array}{r} 18 \\ \underline{4} \end{array}$

2. $\begin{array}{r} 22 \\ \underline{5} \end{array}$

3. $\begin{array}{r} 27 \\ \underline{8} \end{array}$

4. $\begin{array}{r} 34 \\ \underline{6} \end{array}$

5. $\begin{array}{r} 43 \\ \underline{5} \end{array}$

6. $\begin{array}{r} 19 \\ \underline{5} \end{array}$

7. $\begin{array}{r} 32 \\ \underline{7} \end{array}$

8. $\begin{array}{r} 44 \\ \underline{6} \end{array}$

9. $\begin{array}{r} 28 \\ \underline{9} \end{array}$

10. $\begin{array}{r} 35 \\ \underline{4} \end{array}$

11. $\begin{array}{r} 63 \\ \underline{13} \end{array}$

12. $\begin{array}{r} 27 \\ \underline{12} \end{array}$

13. $\begin{array}{r} 48 \\ \underline{23} \end{array}$

14. $\begin{array}{r} 56 \\ \underline{17} \end{array}$

15. $\begin{array}{r} 47 \\ \underline{18} \end{array}$

16. $\begin{array}{r} 221 \\ \underline{19} \end{array}$

17. $\begin{array}{r} 532 \\ \underline{56} \end{array}$

18. $\begin{array}{r} 818 \\ \underline{35} \end{array}$

19. $\begin{array}{r} 761 \\ \underline{44} \end{array}$

20. $\begin{array}{r} 542 \\ \underline{47} \end{array}$

21. $\begin{array}{r} 815 \\ \underline{76} \end{array}$

22. $\begin{array}{r} 614 \\ \underline{60} \end{array}$

23. $\begin{array}{r} 453 \\ \underline{80} \end{array}$

24. $\begin{array}{r} 371 \\ \underline{90} \end{array}$

25. $\begin{array}{r} 109 \\ \underline{75} \end{array}$

26. $\begin{array}{r} 601 \\ \underline{48} \end{array}$

27. $\begin{array}{r} \$19.77 \\ \underline{402} \end{array}$

28. $\begin{array}{r} \$42.29 \\ \underline{803} \end{array}$

B. Find the quotients.

1. $165 \div 3$
2. $348 \div 6$
3. $144 \div 12$
4. $343 \div 7$
5. $168 \div 8$
6. $495 \div 15$
7. $288 \div 9$
8. $224 \div 8$
9. $102 \div 6$
10. $322 \div 7$
11. $192 \div 12$
12. $363 \div 11$
13. $351 \div 9$
14. $216 \div 8$
15. $576 \div 9$
16. $572 \div 11$
17. $4116 \div 42$
18. $4375 \div 35$
19. $1536 \div 48$
20. $3456 \div 72$
21. $6120 \div 85$
22. $12{,}348 \div 196$
23. $11{,}664 \div 216$
24. $13{,}038 \div 159$

If you need practice, use the Practice Exercises on page 91. If not, you may work in the Experts' Corner on pages 92 and 93.

A. *Multiplication*

EXAMPLES

1. First write a 0 in the product for the 0 in the multiplier. Why?
Then find the product: 7×516
Write this product to the left of the 0. Why?

$$\begin{array}{r} 516 \\ \underline{70} \\ 36120 \end{array}$$

2. First write the product: $6 \times \underline{?}$
Then find the product: $300 \times \underline{?}$.
Then add the two products.
The sum is the answer.

$$\begin{array}{r} 527 \\ \underline{306} \\ 3162 \\ \underline{15810} \\ 161262 \end{array}$$

Multiply:

1. $\begin{array}{r} 241 \\ \underline{40} \end{array}$
2. $\begin{array}{r} 312 \\ \underline{50} \end{array}$
3. $\begin{array}{r} 151 \\ \underline{60} \end{array}$
4. $\begin{array}{r} 215 \\ \underline{20} \end{array}$
5. $\begin{array}{r} 342 \\ \underline{30} \end{array}$
6. $\begin{array}{r} 212 \\ \underline{50} \end{array}$
7. $\begin{array}{r} 413 \\ \underline{70} \end{array}$
8. $\begin{array}{r} \$51.05 \\ \underline{602} \end{array}$
9. $\begin{array}{r} \$81.90 \\ \underline{305} \end{array}$
10. $\begin{array}{r} \$90.80 \\ \underline{150} \end{array}$
11. $\begin{array}{r} \$46.00 \\ \underline{704} \end{array}$
12. $\begin{array}{r} \$18.05 \\ \underline{308} \end{array}$

B. *Long division, whole numbers*

EXAMPLE

Use 4 as the trial divisor.
In 9 there are 2 fours, so write 2 above the 6 in 9635.
Multiply 41×200 and write the result below the 9635.
Subtract 8200 from 9635 and the remainder is 1435.
You now have a new dividend 1435.
Now, think "41 goes into 1435."
Finish the explanation.

$$\begin{array}{r} 235 \\ 41\overline{)9635} \\ \underline{8200} \\ 1435 \\ \underline{1230} \\ 205 \\ \underline{205} \end{array}$$

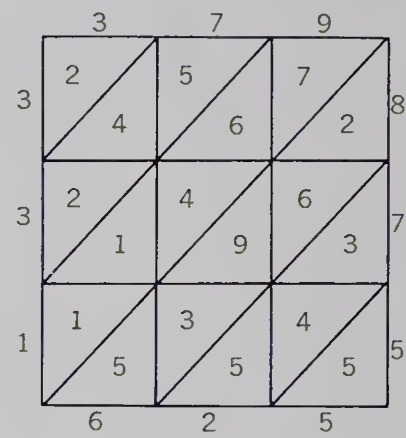
Find the quotients.

1. $1512 \div 72$
2. $1632 \div 17$
3. $4464 \div 72$
4. $3045 \div 87$
5. $8272 \div 47$
6. $5544 \div 56$
7. $5472 \div 76$
8. $11,524 \div 86$
9. $9648 \div 48$
10. $96,888 \div 44$
11. $32,164 \div 43$
12. $438,278 \div 62$

Some Other Ways of Multiplying

A popular method for arranging a multiplication exercise during the eighteenth century was the "lattice" plan.

The lattice arrangement for multiplying 379 by 875 is shown here. Note that one of the factors is at the top, and the other at the right side. Here are the steps in the multiplication:



Step a. Begin multiplying the number named by the digit on the right (in this case 8) by each of those across the top: 3, 7, and 9. Place the products in the squares directly across from the 8 and directly under the number at the top which was multiplied. The product of 8 and 3 is 24 and is placed directly under the 3 across from 8.

Step b. You see that each cell in the lattice is divided by a diagonal line. The products are placed so that the digit in the ones place of a product is located in the lower right half of the cell and tens digit is in the upper left half of the cell. When the product of two numbers produces only a one-digit numeral (such as 2×3) the upper left half of the cell can be filled with zero.

Step c. Multiply each of the numbers, 8, 7, and 5 by 3, 7, and 9, and place the products in the appropriate cell.

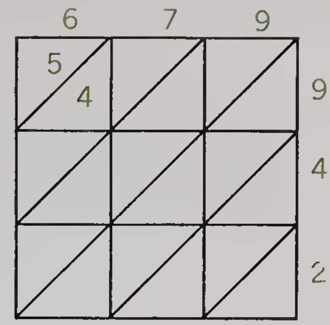
$8 \times 9 = 72$
$8 \times 7 = 56$
$8 \times 3 = 24$
Check the values in the lattice.

Step d. To total the partial products, add diagonally the members of each parallel section beginning at the lower right. Place the totals at the bottom of the square and up the left side, as shown. Carry any excess values into the next cell as you do in ordinary addition.

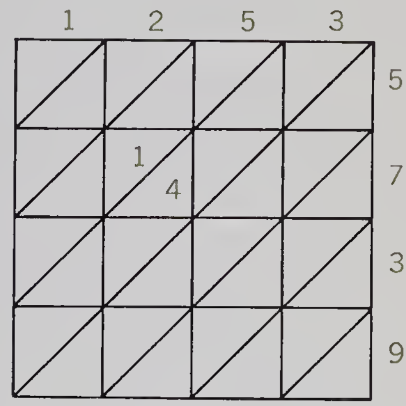
If you read the numbers named on the left and bottom sides of the lattice in order, they are 3, 3, 1, 6, 2, and 5. If you multiply 379 by 875 using ordinary arithmetic, the result is 331,625. This is the number named by the digits on the left and bottom sides of the lattice.

Review the steps in the lattice arrangement of multiplication before doing the Exercises on page 93.

1. Here is the lattice arrangement for finding the product of 679 × 942. 9 × 6 is already done. Copy and complete the lattice.



2. At the right below is the lattice arrangement to find the product of 1253 × 5738. Copy and complete.



Use the lattice arrangement for finding:

3. 126 × 492
4. 837 × 168
5. 644 × 525
6. 1752 × 45
7. 2921 × 58
8. 2927 × 69
9. 4392 × 136
10. 7751 × 278

The Russian peasant could not find the product for two numbers such as 38 and 33 by our method of multiplication, but he could work problems using the number two. For example, he could multiply and divide by using 2. Therefore, it is said that he developed the method of multiplication, using 2.

EXAMPLE

What is the product of 38 and 33?

Solution: In a table, place 33 on the right side and 38 on the left. Multiply the right column by 2 and divide the left by 2. Discard all remainders. The last quotient will always be 1. Cross out each number named in the right column where the opposite number named in the left column is even. Then add the remaining numbers named in the right column. The result is the product you want.

L	*R*
38	~~33~~
19	66
9	132
4	~~264~~
2	~~528~~
1	1056
	1254 — Answer

11. Although it does not matter which number is named in the left and which is in the right column, the work is shorter with the smaller number on the left. Try the above computation with 33 on the left. Why is the work shorter when the smaller number is on the left?

Use the Russian Peasant Method for finding these products: (Check your work using ordinary arithmetic.)

12. 46 × 28
13. 54 × 19
14. 27 × 36
15. 32 × 49
16. 58 × 27
17. 65 × 14

In multiplication you have a certain number of sets, each having the same number of elements. The operation is to combine them into a larger set. Thus:

x	$\times$	y	$=$	p
number of sets		number of elements per set		larger set

As we have seen, division is the inverse of multiplication. In a division situation you are given the product and one of the factors. The given factor may be the number of sets, or the number of elements in each set. You are to find the other factor. Let us look at an example of each kind of situation.

EXAMPLES

1. (You are to find the number of elements in a set.)
   The committee planning the class picnic purchased 8 dozen buns. There will be 32 persons at the picnic. How many buns did they purchase for each person?
   Statement: $96 \div 32 = n$

2. (You are to find the number of sets.)
   The planning committee prepared a 3-inch badge for each person attending the picnic. They were cut from a 96-inch length of ribbon. How many badges could they cut from the ribbon?
   Statement: $96 \div 3 = n$

The following problems call for division. If you examine each one carefully you can identify whether you are finding the number of sets or the number of elements in a set.

1. The Bradleys are planning a trip of 960 miles in their car. The car averages about 16 miles to a gallon of gasoline. How many gallons will they use on the trip?

2. Mike weighs 140 pounds. His dog weighs 35 pounds. Mike weighs how many times as much as his dog?

3. Ned purchased a boat for \$265. He paid \$45 in cash, and the rest in 11 equal monthly payments. How much was each payment?

4. The legal speed limit for automobiles in many states is 65 miles per hour. A jet plane is scheduled to travel from Los Angeles to New York at 585 miles per hour. This is how many times the legal speed limit for an automobile?

You are *not* to work these problems, but to answer questions about them. Write the numerals 1 through 7 on your paper. After each problem are some possible answers. You are to select the best answer to complete the statement. Write the letter to indicate that answer after the proper numeral on your paper. Be sure to read each problem and answer carefully.

1. Mrs. Jones bought three crates of strawberries to preserve, at \$2.50 a crate, and 20 pounds of sugar at 5 pounds for 39 cents. She paid for this with a \$10 bill. The problem tells me that:

a. Mrs. Jones paid 39 cents a pound for strawberries.
b. Mrs. Jones had a ten-dollar bill.
c. The strawberries sold at five pounds for \$2.50.
d. Mrs. Jones used 20 pounds of sugar to can strawberries.

2. Martin's allowance is \$3.50 a week. Every day he spends 45 cents for lunch. One week he also spent 65 cents for the movies and 20 cents for a soda. This problem tells me that:

a. Martin spent 20 cents for lunches and 35 cents for movies.
b. Martin's allowance is \$3.50 a week.
c. Martin saved 65 cents out of his allowance.
d. His regular expenses for each week were \$3.50.

3. Jim works in a grocery store two hours after each school day and eight hours on Saturday. On school days he begins work at 4:00 P.M. and on Saturdays at 9:00 A.M. He is paid 95 cents an hour. The fact I do *not* need in order to find how much Jim earns during each week is:

a. The time he begins work on school days.
b. The amount he makes each hour.
c. The number of hours he works on Saturdays.
d. The number of hours he works each school day.

4. Mr. Adams used 10 gallons of oil in his car each year. He can get the oil in five-gallon cans at \$4.80 per can. The same oil when purchased by the quart costs him 45 cents a quart. The fact I do *not* need in order to find the saving on each quart in buying 5-gallon cans is:

a. The number of gallons of oil Mr. Adams uses a year.
b. The cost of a quart of oil bought by the quart.
c. The cost of a five-gallon can of oil.
d. The cost of a quart of oil bought in a five-gallon can.

5. Billy has 36 regular customers on his paper route. He starts delivery at 4:30 P.M. every day after school. On Mondays he collects the weekly bill of 42 cents from each customer. The fact I do *not* use in calculating Billy's weekly collection is:

a. The time he starts delivering papers.
b. The number of customers he has.
c. The amount he collects from each customer for the week.
d. The number of papers delivered regularly each day.

6. Jane earns $270 a month. She works six days a week, beginning at 8:00 A.M. and finishing at 4:30 P.M., except on Saturday when she is through at noon. She works 12 months of the year. The fact I do *not* use to find Jane's annual salary is:

a. The number of months she works in a year.
b. The time she begins work each day.
c. The salary she earns each month.

7. Jane, Mary, Ruth, and Betty gave a party together and shared expenses. They served ham, eggs, peanut butter sandwiches, cookies, punch, and ice cream. Their grocery bill came to $13.25 without the ice cream. The ice cream cost 79 cents a half-gallon and they needed six quarts. In figuring each girl's expenses I do *not* use:

a. The amount of ice cream needed.
b. The cost of groceries.
c. The price of the ice cream.
d. The menu for the party.

8. Harry purchased a used outboard motor. He paid $75 in cash, and the rest in equal monthly payments of $10. In order to find the total amount Harry paid for the motor I need to know:

a. What the motor cost when new.
b. How long it had been used.
c. The number of monthly payments.
d. Whether or not Harry has a job.

9. Last summer the Jones family took a vacation trip in their car; it lasted about two weeks. They used 150 gallons of gasoline on the trip. To find the distance they traveled, on the average, per gallon, you need to know:

a. The cost of gasoline per gallon.
b. The average speed of driving.
c. The number of hours they traveled per day.
d. The total number of miles they traveled.

1. A truck driver made a trip of 440 miles in 8 hours. What was the average distance per hour he traveled?
2. An orchard contains 40 rows of trees, with 24 trees in a row. How many trees are in the orchard?
3. A truckload of potatoes contains 3600 pounds of potatoes. There are 60 pounds of potatoes in a bushel. How many bushels of potatoes are there on the truck?
4. Henry worked 38 hours at the filling station last week at $1.05 an hour. How much did he earn?
5. A car traveling 48 miles an hour is using a gallon of gasoline each 16 miles. How many gallons will it use in 4 hours?
6. Jane earned $7.60 in 8 hours typing manuscripts last Saturday. This is at the rate of how much per hour?
7. Mike earns $1.15 an hour at the supermarket working after school and Saturdays. He plans to purchase a bicycle that will cost $46. How many hours must he work to pay for the bicycle?
8. Mr. Henderson is planning a trip of 900 miles in his car. The car averages 15 miles to the gallon of gasoline. How many gallons of gasoline should Mr. Henderson expect to use on the trip?
9. Jim purchased a set of golf clubs for $65. He paid $25 down and the rest in 8 equal monthly payments. How much was each payment?
10. On their vacation trip last summer the Andersons traveled 1050 miles, and used 70 gallons of gasoline. How many miles per gallon did the car travel, on the average?
11. A brick is 2 inches thick. How many feet high is a stack of 48 bricks?
12. A box of books weighs 45 pounds. Each book weighs 20 ounces. The box without the books weighs 5 pounds. How many books are in the box? (1 pound is 16 ounces.)
13. When John's older brother graduated from college he was employed at $450 a month. This is 3 times what John earns per month during summer vacation. How much does John earn per month?
14. Miss Adams says that those students whose average is 93 or better will be given top grades for the month. Henry's grades for the 4 weeks have been: 89, 96, 94, and 93. Did he qualify for the top grade?
15. Mr. Anderson delivered 3000 pounds of wheat. A bushel of wheat weighs 60 pounds. How many bushels did he deliver?

In solving the following conditional statements, first perform whatever additions or other operations are necessary so that you have not more than one number named on each side of the equation. Then write two equivalent statements, and find the value of the variable, using the form that has the variable alone on one side.

EXAMPLES

1.

	$y + 17 - 4 = 25 + 11$	
	$y + 13 = 36$	$a + b = s$
Then	$13 = 36 - y$	$b = s - a$
and	$y = 36 - 13$	$a = s - b$
so	$y = 23$	

2.

	$n \div 8 = 23 - 16$	
	$n \div 8 = 7$	$p \div x = y$
Then	$n \div 7 = 8$	$p \div y = x$
and	$n = 7 \times 8$	$p = x \times y$
so	$n = 56$	

1. $x - 16 = 27$

2. $3y = 36$

3. $x \div 7 = 21$

4. $49 - n = 38$

5. $4y = 56$

6. $36 \div x = 4$

7. $n + 35 = 50$

8. $111 \div y = 37$

9. $108 = 12x$

10. $29 - y = 23 - 10$

11. $8 + n = 11 + 7$

12. $6y = 21 + 15$

13. $48 = x + 13 + 15$

14. $12 + 13 = 5n$

15. $y = 13 + 15 + 23$

16. $(25 + 15) \div n = 5$

17. $x + 24 = 12 + 14 + 5$

18. $112 - y = 11 + 5$

19. $9n = 45 + 28 + 35$

20. $18x = 50 + 22 + 18$

21. $x + 15 = 63 + 22$

22. $54 \div x = 9$

23. $115 - y = 14 + 30$

24. $8n + 3 = 27$

25. $4 = (15 + 9) \div x$

26. $84 = 5x + 9$

27. $36 + 8 = 18 + y$

28. $n \div 14 = 5 - 3$

29. $28 + 3 = 31 - y$

30. $63 \div n = 1$

31. $(14 + 6) \div 20 = x$

32. $26 + y = 31 - 5$

33. $7 \times (20 + 2) = y$

34. $(46 + 6) \div n = 13$

35. $15 \times (2 + x) = 45$

36. $n \div 56 = 0$

Use mathematical symbols and write a conditional statement that expresses each of the following sentences. Then solve, and check your answer by substituting the value for the variable in the original sentence.

1. If 14 is added to a number the sum is 37.
2. Three times a certain number is 57.
3. A certain number divided by 13 is 117.
4. If a certain number is divided by 7 the quotient is 21.
5. Twenty-five less than a certain number is 33.
6. Thirteen more than a certain number is 30.
7. If a certain number is subtracted from 49 the difference is 8.
8. Fifty-six is 8 times a certain number.
9. If 222 is divided by a certain number the quotient is 74.
10. If a certain number is subtracted from 39 the result is 10 less than 28.
11. Eighteen more than a certain number is 10 more than 17.
12. One hundred twenty-eight is 16 times a certain number.
13. Eighty-five divided by a certain number is 9 less than 26.
14. Nine times a certain number is 11 less than 110.
15. Eighty-one divided by a certain number is equal to 36 divided by 4.
16. Eight times a certain number is 256.
17. Three hundred forty-three divided by a certain number is 7.
18. If a certain number is added to 17 the result is 4 times 10.
19. The sum of a certain number and 35 is 50.
20. If a certain number is added to 25 the sum is the square of 7.
21. A certain number is the sum of 18 and 19.
22. A certain number multiplied by 8 is 128.
23. If 306 is divided by a certain number the quotient is 34.
24. A certain number is equal to the product of 9 and 12.
25. If a certain number is multiplied by 6, the result is 5 times 12.
26. A certain number is equal to the difference between 23 and 15.
27. The quotient of 36 and a certain number is 2.
28. A certain number is equal to the sum of 19 and 32 decreased by 7.
29. The square of a certain number is 81.
30. A certain number is equal to the square of 24.

Zero as a number is a rather recent discovery. We saw earlier that the Romans did not have a numeral for zero. However, the idea of zero has always existed. Zero is the only number in the set of whole numbers that is not in the set of natural, or counting, numbers. There are some other special properties of zero which may be discovered by working the exercises below.

1. Add in each of the following:

a. $2 + 0$	**c.** $0 + 16$	**e.** $0 + 0$
b. $7 + 0$	**d.** $0 + 85$	**f.** $316 + 0$

2. State a rule for adding zero to any whole number.

We call zero the *identity element* for addition, or the *additive identity,* because when we add zero to any whole number or add any whole number to zero, we get *identically* that same number.

3. Subtract in each of the following:

a. $6 - 0$	**c.** $126 - 0$	**e.** $312 - 0$
b. $4 - 0$	**d.** $0 - 0$	**f.** $624 - 0$

4. State a rule for subtracting zero from any whole number.

5. Subtract in each of the following:

a. $8 - 8$	**c.** $81 - 81$	**e.** $0 - 0$
b. $6 - 6$	**d.** $65 - 65$	**f.** $31{,}416 - 31{,}416$

6. State a rule for subtracting a whole number from itself.

7. Multiply in each of the following:

a. 6×0	**c.** 0×7	**e.** 0×0
b. 11×0	**d.** 0×42	**f.** 116×0

8. State a rule for multiplying a whole number by zero.

9. When we divide zero by a natural number, as $0 \div 5$, we are asking to find n so that:

$$0 \div 5 = n.$$

If $0 \div 5 = n$ is a true statement, then: $0 = 5 \times n$. This can be a true statement only if $n = 0$. Why?

10. Then zero divided by a natural number gives the quotient zero. Divide the following:

a. $0 \div 4$	**c.** $0 \div 3$	**e.** $0 \div 17$
b. $0 \div 7$	**d.** $0 \div 12$	**f.** $0 \div 31$

11. To divide a natural number, such as 5, by zero, you must find n so that

$$5 \div 0 = n.$$

If this is a true statement, then $5 = n \times 0$. Since the product of *any* natural number and zero is zero, you cannot find a value for n to fit these conditions. Therefore, *you cannot divide a natural number by zero.* Write the above proof using some number other than 5. Similarly, if $0 \div 0 = n$, where n is a whole number, then $0 = n \times 0$. However, the product of *any* whole number and zero is zero. Therefore, $0 \div 0$ is meaningless.

12. If n represents any whole number, replace the question mark in each statement by the value that makes the statement true.

a. $n + 0 = ?$ **b.** $n - 0 = ?$ **c.** $n - n = ?$ **d.** $n \times 0 = ?$

13. If n represents any natural number, then: $0 \div n = ?$

14. Find the value for y to make each a true statement.

a. $4 - 4 = y$	**f.** $86 + 0 = y$	**k.** $16 + 0 = y$
b. $8 + 0 = y$	**g.** $31 \times 0 = y$	**l.** $0 \div 13 = y$
c. $11 \times 0 = y$	**h.** $0 \div 5 = y$	**m.** $4 - 0 = y$
d. $0 + 144 = y$	**i.** $0 \times 16 = y$	**n.** $8 - 8 = y$
e. $91 - 91 = y$	**j.** $7 - 0 = y$	**o.** $(0 \div 3) + (6 - 6) = y$

Let us summarize some of the important ideas we have discussed concerning zero.

When we add or subtract zero from any whole number, the result is the same whole number. In symbols,

$$n + 0 = n \qquad \text{and} \qquad n - 0 = n$$

where n represents any whole number. Also, when we subtract any whole number from itself, the result is zero. In symbols,

$$n - n = 0$$

where n represents any whole number.

When zero is multiplied by any whole number, the result is zero. In symbols,

$$n \times 0 = 0$$

where n represents any whole number. Also when zero is divided by a natural number, the result is zero. In symbols,

$$0 \div n = 0$$

where n represents a natural number. Remember! Division by zero is meaningless.

The number "one" is less than any other natural number. Like zero, it also has special properties.

1. Subtract in each of the following:

a. $8 - 7$ **b.** $1 - 0$ **c.** $116 - 115$ **d.** $47 - 46$

2. State a rule for the difference between any two consecutive whole numbers.

3. Multiply in each of the following:

a. 1×4 **b.** 6×1 **c.** 1×0 **d.** 67×1

4. State a rule for multiplying any whole number by one.

5. Divide in each of the following:

a. $7 \div 1$ **b.** $212 \div 1$ **c.** $0 \div 1$ **d.** $81 \div 1$

6. State a rule for dividing any whole number by one.

7. Divide in each of the following:

a. $17 \div 17$ **b.** $1 \div 1$ **c.** $167 \div 167$ **d.** $8 \div 8$

8. State a rule for dividing a natural number by itself. Why do we state this rule for natural numbers and not whole numbers?

9. Perform the following operations.

a. $15 - 14$ **e.** $13 \div 1$ **i.** $146 \div 1$
b. $14 + 1$ **f.** $77 - 76$ **j.** $164 \div 164$
c. 1×9 **g.** $76 + 1$ **k.** 67×1
d. $11 \div 11$ **h.** 99×1 **l.** $(5 \times 5) \div (10 + 15)$

HINT: In l, remember to do the work in the parentheses first.

10. Using the rules you have developed, complete the following:

a. $n \times 1 = ?$ **b.** $1 \times n = ?$ **c.** $n \div 1 = ?$ **d.** $n \div n = ?$

11. The number one is called the *identity element* for multiplication or *multiplicative identity*, because when we multiply any number by one or multiply one by any number, we get *identically* that same number. Which of the statements in Exercise 10 shows this? Which of the Exercises above illustrates this?

12. Make each of the following statements true by replacing the ○ with the symbol for *equals* (=), *is greater than* (>), or *is less than* (<). Remember! Do the work in the parentheses *first*.

a. $143 - 142$ ○ 1 **d.** $(14 \times 1) - 13$ ○ 1
b. $(16 - 15) + (14 - 14)$ ○ 1 **e.** $(7 \times 7) \div 49$ ○ 1
c. $(13 \div 13) + (12 \div 12)$ ○ 1 **f.** $(2 \times 3) \div (3 \times 2)$ ○ 1

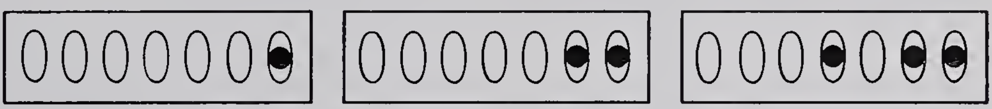
The Binary System

The numeration system with base two is called the *binary system.*

Just as a number system with base ten has ten symbols, a system with base two has two symbols, 0 and 1. High speed digital computers represent numbers in the binary system by using a series of electric switches. The computer represents the symbol 1 by a switch turned on, and 0 by a switch turned off.

The binary system can be represented on a sand table by making a series of parallel grooves each only large enough to hold one pebble. The grooves represent the places in the numeration system. We use the places ones, tens, hundreds, etc., in our base-ten system; however, the places (grooves) in this sand table would have different names. We can represent one by placing a pebble in the first groove. This place is now full. To represent two (base ten) we would place the pebble in the next groove to the left. This would represent one 2 and no ones. The grooves act like switches on the computer. When a groove has a pebble in it, the place is filled (similar to a closed switch).

The sketches below represent numerals. We can write the base-two numeral represented in each sketch by writing a 1 for each groove with a pebble in it and a 0 for each groove that is empty.



a. b. c.

The base-two numerals represented in the sand tables above are 1, 11, and 1011, respectively. What do they mean in base ten?

1. In our base-ten system, the value of each place is multiplied by 10 as we go from right to left. In base five, we saw that the value of each place was multiplied by 5 as we went from right to left. In the binary system, the value of each place should be multiplied by what factor as we go from right to left?

2. Since any number raised to the zero power is 1, the first groove on the right in the sand table can be represented as 2^0, or 1. The next groove to the left then is 2^1, or 2. The third groove is 2^2, or 4. What is the value of each of the next four grooves?

3. Sand table (a) above names the number 1 in base ten. Sand table (b) names the number 3 in base ten since there is one pebble in the groove whose value is 2^1, or 2, and one pebble in the groove whose value is 1. Then: 11 base two = 3 base ten.

Let us analyze 1011 base two, moving from right to left.

Base two		*Base ten*
$1 = 1 \times 2^0 =$	$1 \times 1 =$	1
$1 = 1 \times 2^1 =$	$1 \times 2 =$	2
$0 = 0 \times 2^2 =$	$0 \times 4 =$	0
$1 = 1 \times 2^3 =$	$1 \times 8 =$	8
		11

Then 1011 base two = 11 base ten

Find the value of each of the following (base two) in base ten.

a. 101 **b.** 111 **c.** 1001 **d.** 1111 **e.** 10000 **f.** 10001

4. To represent 4 base ten on the sand table, we would place one pebble in the groove whose value is 2^2, or 4. To represent 8 base ten, we would place one pebble in the groove whose value is 2^3, or 8. To represent 5 base ten, we would place one pebble in the groove whose value is 2^2, or 4, and one pebble in the groove whose value is 2^0, or 1. Draw a sketch of a sand table to show this. Write the base-two numeral that your sketch represents.

5. To represent 17 base ten on the sand table, we would place one pebble in the groove whose value is 16 (2^4) and one pebble in the groove whose value is 1 (2^0). To represent 25 base ten, we would place a pebble in the grooves whose values are 16 (2^4), 8 (2^3), and 1 (2^0). Draw sketches of a binary sand table to show these. Write the base-two numerals that your sketches represent.

6. As your answers to Exercises 4 and 5 show, 5 base ten = 101 base two, 17 base ten = 10001 base two, and 25 base ten = 11001 base two. Here are the steps for writing base-ten numerals in base two.

a. Express the base-ten numeral as a sum such that each number named in the sum is the greatest possible power of 2.

b. Replace each number named in the sum by the base-two number it names. Add the base-two numbers named in the sum.

EXAMPLES

1. 33 base ten = _?_ base two

$$33 = 32 + 1$$
$$= 2^5 + 2^0$$
$$= (100000 + 1) \text{ base two}$$
$$= 100001 \text{ base two}$$

33 base ten = 100001 base two

2. 46 base ten = _?_ base two

$$46 = 32 + 8 + 4 + 2$$
$$= 2^5 + 2^3 + 2^2 + 2^1$$
$$= (100000+1000+100+10) \text{ base two}$$
$$= 101110 \text{ base two}$$

46 base ten = 101110 base two

Write base-two numerals for the following base-ten numerals:

a. 9 **b.** 12 **c.** 18 **d.** 37 **e.** 55 **f.** 70

Part One

A. Do not copy. Place a sheet of paper under each column and add.

1.	2.	3.	4.	5.	6.
75	35	57	963	43	3995
59	68	29	301	416	3
		62	360	2010	1785
		76	275	109	98
		89	908	319	20

B. Do not copy. Place a sheet of paper under each exercise and subtract.

1.	3.	5.	7.
37	42	580	550
23	26	376	407
2.	**4.**	**6.**	**8.**
55	406	502	800
32	322	344	747

C. Find the products.

1.	4.	7.	10.
221	751	814	109
19	44	70	96
2.	**5.**	**8.**	**11.**
532	542	453	601
56	47	60	48
3.	**6.**	**9.**	**12.**
818	815	371	356
35	75	80	39

D. Find the quotients.

1. $2257 \div 61$
2. $9936 \div 48$
3. $1620 \div 54$
4. $8265 \div 95$
5. $4532 \div 44$
6. $10{,}836 \div 252$
7. $1260 \div 36$
8. $3234 \div 42$
9. $2952 \div 72$
10. $5544 \div 66$
11. $22{,}032 \div 24$
12. $22{,}320 \div 124$

Part Two

A. Find the value for N that makes each a true statement.

1. $16 \times 33 = N$
2. $N \times 26 = 728$
3. $45 \times N = 1035$
4. $42 \times 67 = N$
5. $N \times 35 = 595$
6. $240 \div N = 15$
7. $273 \div N = 7$
8. $N \div 25 = 625$
9. $700 \div 25 = N$

B. Find the answer to each of the following:

1. $134 + 0$	**4.** $0 \div 16$	**7.** $93 - 93$
2. 42×0	**5.** $0 + 67$	**8.** $(0 \div 3) + (14 - 4)$
3. $19 - 0$	**6.** 0×95	**9.** $6 \times 7 \times 2 \times 0$

C. List the numerals 1 through 10 on your paper. After each numeral place a letter to indicate which of the expressions listed on the right is defined or illustrated by each statement on the left having that numeral.

1. the inverse operation of multiplication	**a.** product
2. factor $\times$ factor = ?	**b.** associative property
3. dividend $\div$ divisor = ?	**c.** commutative property
4. product $\div$ factor = ?	**d.** division
5. identity element for multiplication	**e.** factor
6. repeated addition	**f.** multiplication
7. $(49 + 8) + 62 = 49 + (8 + 62)$	**g.** one
8. $325 \times 6 = 6 \times 325$	**h.** quotient
9. the numbers added to form a sum	**i.** zero
10. identity element for addition	**j.** addends

D. List the numerals 1 through 10 on a sheet of paper. Study each of the following statements, and write the letter to indicate your choice after the numeral of each statement.

1. In multiplication, $x \times y$ equals the:

a. sum **b.** product **c.** quotient **d.** difference

2. In a division operation, the dividend should equal the divisor times the quotient plus the:

a. addend **b.** difference **c.** remainder **d.** minuend

3. The identity element for addition is:

a. one **b.** zero **c.** ten **d.** two

4. The identity element for multiplication is:

a. one **b.** zero **c.** ten **d.** two

5. If a division operation does not "come out even" the number left over is called the:

a. difference **b.** remainder **c.** divisor **d.** dividend

6. The only whole number that is not a counting number is:

a. 1 **b.** 0 **c.** 10 **d.** 100

7. If $n \div 17$ is equal to $17 \div n$, then the value of n is:

a. 17 b. 12 c. 1 d. 0

8. The commutative property holds for:

a. multiplication and division
b. multiplication and addition
c. addition and subtraction
d. multiplication and subtraction

9. To multiply 32×125 without pencil and paper you could multiply:

a. $4 \times (8 \times 125)$
b. $30 \times (2 \times 125)$
c. $(32 \times 100) \times 25$
d. $(32 \times 25) \times 100$

10. The division operation can be thought of as:

a. repeated addition
b. repeated subtraction
c. the inverse of addition
d. the inverse of subtraction

Part Three

1. After driving his car for 10,000 miles, Martin calculated that it cost him 6¢ a mile to operate his car. At that rate how much would it cost him to travel 870 miles?
2. A cash register contained $513.43. The cashier added $11.28, $7.11, and $3.26 to what was there and then removed $400 to deposit in the bank. How much was left in the cash register?
3. Jane works as usher in a theater at $1.10 an hour, for 24 hours a week. Marian works as secretary 20 hours a week at $1.25 an hour. Who earns more during the week, and how much?
4. Joe can run 100 yards in 10 seconds. At that rate how long should it take him to run 20 yards?
5. An airplane is traveling at 575 miles an hour. How far can it travel in 4 hours at that rate?
6. Mr. Hansen shipped a truckload of hogs to Jamestown. He paid 25¢ per 100 pounds for shipping. The total cost of shipping was $72. What was the weight of the hogs?
7. James and his father made a trip of 810 miles in their car. They averaged 15 miles to the gallon of gasoline. At 34¢ per gallon, how much did the gasoline cost for the trip?
8. A moving van is transporting furniture to a city 2350 miles away. The driver expects to make the trip in 5 days. How many miles per day must he average?
9. Mr. Britten estimates that it costs him 11¢ a mile to operate his car. He has driven it 52,695 miles. How much did it cost him to drive that distance?

Part One

Add:

1. 25	**2.** 216	**3.** 4625	**4.** \$114.49
89	457	381	25.36
56	646	7946	62.55
77	983	537	7.18
45	435	829	32.96

Part Two

Subtract:

1. 364	**2.** 506	**3.** 753	**4.** 8734
− 195	− 238	− 404	− 6175

Part Three

Multiply:

1. 42	**2.** 37	**3.** 59	**4.** 307
4	6	8	280

Divide: Find the quotients to the nearest cent.

5. \$792.45 ÷ 95 **7.** \$856.39 ÷ 271 **9.** \$656.83 ÷ 191

6. \$325.50 ÷ 77 **8.** \$617.20 ÷ 116 **10.** \$392.87 ÷ 301

Part Four

1. Write the Arabic numerals for:

a. MC **b.** XXXVII **c.** CDXLIV **d.** MDC

2. Write the Roman numerals for:

a. 35 **b.** 792 **c.** 1965 **d.** 1066

3. Change each of the following from base ten to base five.

a. 6 **b.** 13 **c.** 27 **d.** 326

4. Change each of the following from base five to base ten.

a. 4 **b.** 12 **c.** 43 **d.** 133

Part Five

Solve for N in each of these conditional statements.

1. $N \div 15 = 6$
2. $N - 14 = 27$
3. $18 \div N = 3$
4. $56 - N = 35$
5. $N \times 25 = 300$
6. $45 + N = 80$
7. $63 \div N = 7$
8. $N + 18 = 29$
9. $72 \div N = 8$
10. $58 + N = 72$
11. $13 \times N = 91$
12. $N - 19 = 36$
13. $17 \times N = 102$
14. $96 - N = 27$
15. $N \times 19 = 95$
16. $N - 46 = 17$
17. $N \div 50 = 1900$
18. $16 + N = 83$
19. $N + 35 = 46$
20. $52 \div N = 13$

Part Six

On a sheet of paper write the numerals 1 through 15. In each of the following problems, before you calculate the precise answer with pencil and paper, first estimate a reasonable answer with rounded numbers. Put this estimate after the numeral for the problem on your paper. Then find the precise answer, and put it alongside your estimate.

1. Betty received 60¢ per hour for baby-sitting. She needed to earn \$7.50 during the month of September to purchase her student organization card. How many hours would she need to work?
2. Jane bought three records for \$1.28 each. How much change should she receive from a \$5 bill?
3. At 95¢ per hour, how long would it take to earn \$38?
4. John sold his 4-H Club steer for 28¢ per pound. The steer weighed 1150 pounds. How much money did John receive?
5. It is 234 miles from Elmtown to Lakeview. The bus makes the trip in 6 hours. What is its average rate in miles per hour?
6. If it averages 40 mph, how long will it take a bus to travel 480 miles from Jamestown to Centerville?
7. What would 33 lbs. of sugar cost at 12¢ per lb.?
8. Miss Jensen's class is planning to visit the city hall. For \$6.00 the bus company will provide a bus for the trip. There are 30 in the class. How much will each have to pay?
9. Mary works 9 hours a week after school as secretary in a real estate office. She earns 85¢ an hour. How much does she earn per week?

CHAPTER FOUR

FRACTIONAL NUMBERS

WORDS TO WATCH FOR

decimal
denominator
equivalent fractions
fraction
fractional number
improper fraction
least common multiple
like fractions
mixed numeral
multiplicative inverse
numerator
proper fraction
relatively prime
scientific notation
simplest form
terms
unlike fractions

In examining the number line whose points are associated with the set of whole numbers, you will notice that there are spaces between the points associated with consecutive numbers. There are no numbers associated with points such as A, B, or C on the number line below. Do we need the numbers that might be associated with these and other points?

1. If any two whole numbers are added, what can you say about their sum? Give five examples.

2. If any two whole numbers are multiplied, what can you say about their product? Illustrate your answer.

3. Locate the quotients for the following divisions on the number line:

a. $6 \div 2$ **b.** $21 \div 7$ **c.** $40 \div 8$ **d.** $25 \div 5$

4. Suppose you are asked to divide the following: $4 \div 3$. The dividend, 4, is not exactly divisible by the divisor, 3. Since the remainder, 1, is less than the divisor, we can express the remainder as follows:

$$\frac{1}{3} \quad \frac{\text{(remainder)}}{\text{(divisor)}}$$

Thus the answer to the problem is: 1 plus a remainder of 1, or

(*a*) $1 + \frac{1}{3}$ or simply $1\frac{1}{3}$ (which is not a whole number).

Also, $4 \div 3$ can be expressed as:

(*b*) $$\frac{4}{3} \quad \frac{\text{(dividend)}}{\text{(divisor)}}$$

The line that separates 4 and 3 indicates division, as does the line that separates 1 and 3 above.

Express each of the following in the form of (*a*) and (*b*) above:

a. $7 \div 3$ **b.** $5 \div 2$ **c.** $9 \div 4$ **d.** $17 \div 5$

5. Draw a number line, and show about where each of the numbers named in Exercise 4 should be located.

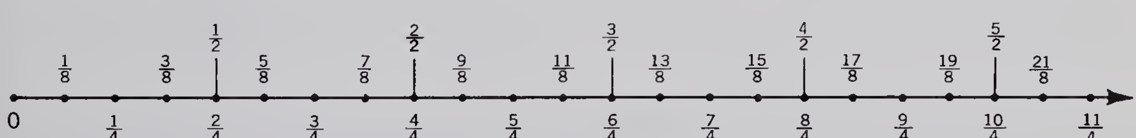
You can see that if we wish to use division as freely as we do addition and multiplication, we must extend the set of numbers to include quotients like those from Exercise 4. These represent the *fractional numbers of arithmetic.* They include such numbers as those named by $\frac{5}{3}$, $\frac{3}{4}$, $\frac{1}{3}$, and $\frac{1}{8}$. It is important to note that these can also be expressed as $5 \div 3$, $3 \div 4$, $1 \div 3$, and $1 \div 8$.

A whole number can also be named by a fraction; that is, 3 can be written as $\frac{3}{1}$, $\frac{6}{2}$, etc. Similarly, 0 can be written as $\frac{0}{1}$, $\frac{0}{2}$, etc. Thus the set of whole numbers is a subset of the set of fractional numbers.

6. A *fraction* is the name for a fractional number. Express the quotients from the following divisions as fractions:

a. $2 \div 3$ **b.** $4 \div 7$ **c.** $15 \div 22$ **d.** $8 \div 3$ **e.** $1 \div 6$

7. The number line below shows the names for some of the numbers associated with points between the whole numbers. Name three numbers represented by points between 0 and $\frac{1}{2}$ on the number line.



8. The symbol $\frac{3}{8}$ tells us that a "whole" has been divided into 8 equal parts, and that we are concerned with 3 of the equal parts. Interpret the fraction $\frac{2}{3}$ in the same manner.

9. In locating $\frac{3}{8}$ on the number line, what is the whole that was divided into 8 equal parts?

10. Can you identify the 3 equal parts of the fraction $\frac{3}{8}$ that you were concerned with?

11. The number named below the line in the fraction is the *denominator*. It indicates, as was noted, the number of parts into which the whole was divided. The number named above the line tells how many of the equal parts are being considered and is called the *numerator*. Locate $\frac{5}{16}$ on the number line, and explain how the denominator is useful in telling how the whole is divided up, and the numerator in telling how many of the equal parts are being considered.

12. The numerator and denominator together are called the *terms* of the fraction. What are the terms of the fraction $\frac{4}{7}$?

13. Using the definition of numerator and denominator, explain the difference between $\frac{1}{2}$ and $\frac{2}{4}$.

14. The numerator of a fraction may be less than, equal to, or greater than the denominator. Examples of fractions whose numerators are less than their denominators are: $\frac{2}{3}$, $\frac{5}{7}$, $\frac{10}{11}$, and $\frac{4}{9}$. Such fractions are called *proper fractions.* Is the value of the numbers named by these fractions greater than 1, or less than 1?

15. Examples of fractions with numerators the same as the denominators are: $\frac{3}{3}$, $\frac{6}{6}$, $\frac{10}{10}$, and $\frac{8}{8}$. Each of these is a name for the same number. What is the number?

16. Examples of fractions whose numerators are greater than their denominators are: $\frac{4}{3}$, $\frac{10}{2}$, $\frac{6}{1}$, $\frac{7}{4}$. These are called *improper fractions.* List five other improper fractions. Is the value of the number named by an improper fraction greater than, equal to, or less than 1?

17. Since $\frac{10}{2}$ means $10 \div 2$, you can see that some improper fractions name whole numbers. What whole number does each of the following name?

a. $\frac{12}{2}$	**b.** $\frac{35}{7}$	**c.** $\frac{42}{6}$	**d.** $\frac{8}{4}$	**e.** $\frac{100}{20}$	**f.** $\frac{39}{13}$

18. State whether each of the following is a proper fraction, another name for 1, or an improper fraction. Rename the improper fractions as whole numbers where possible.

a. $\frac{3}{8}$	**c.** $\frac{3}{2}$	**e.** $\frac{9}{22}$	**g.** $\frac{24}{6}$	**i.** $\frac{6}{7}$
b. $\frac{15}{16}$	**d.** $\frac{12}{3}$	**f.** $\frac{31}{31}$	**h.** $\frac{1}{2}$	**j.** $\frac{7}{6}$

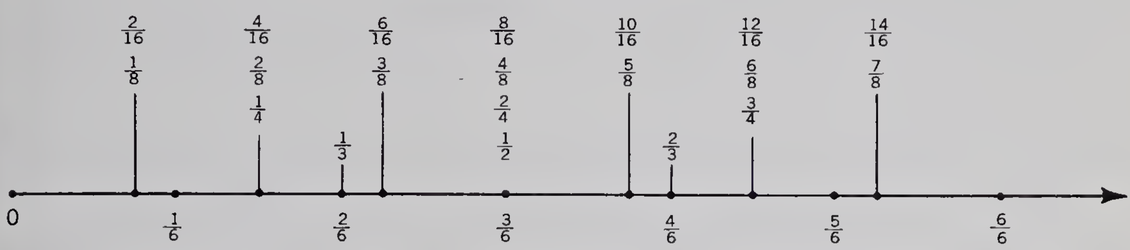
19. Tell whether the members of each of the following sets are less than one, equal to one, or greater than one.

a. The set of proper fractions

b. The set of improper fractions

c. The set of fractions whose numerators are the same as the denominators

Just as there are many names for any whole number, so also there are many names for each of the fractional numbers. The number line below shows that the number represented at the point $\frac{1}{2}$ can also be named $\frac{2}{4}$, $\frac{4}{8}$, $\frac{8}{16}$, or $\frac{3}{6}$. Since these fractions are all names for the same number, they are called *equivalent fractions.*



1. What are two other names for the number associated with the point $\frac{3}{4}$ on the number line?
2. Write two other names for the number associated with the point $\frac{2}{3}$ on the number line.
3. Name two other fractions equivalent to $\frac{1}{8}$.
4. What fraction with denominator 6 is equivalent to $\frac{2}{3}$?
5. What fraction with numerator 6 is equivalent to $\frac{3}{5}$?
6. You will recognize $\frac{1}{2}$ and $\frac{3}{6}$ as equivalent fractions. Notice that the numerator of $\frac{3}{6}$ is 3 times the numerator of $\frac{1}{2}$. Then the denominator of $\frac{3}{6}$ must likewise be how many times as great as the denominator of $\frac{1}{2}$?
7. Multiply the numerator and denominator of $\frac{1}{4}$ each by 2 to obtain a new fraction. Is the new fraction equivalent to $\frac{1}{4}$? Check this on the number line.
8. Multiply the numerator and denominator of $\frac{2}{3}$ each by 3 to obtain a new fraction. Is the new fraction equivalent to $\frac{2}{3}$? Check this on the number line.
9. When you multiplied the numerator and denominator of $\frac{2}{3}$ each by 3, you actually multiplied the fractional number by 1 because $\frac{2 \times 3}{3 \times 3} = \frac{2}{3} \times \frac{3}{3}$. Is $\frac{2}{3}$ equivalent to $\frac{12}{18}$? Why?
10. Find the fraction equivalent to each of the following by multiplying the numerator and denominator by 4.

Rule: To obtain a fraction equivalent to a given fraction, we may multiply the numerator and denominator by the same number.

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a. $\frac{2}{3}$ **b.** $\frac{1}{8}$ **c.** $\frac{1}{10}$ **d.** $\frac{8}{3}$ **e.** $\frac{1}{25}$

The rule in Exercise 10 is true because in so doing, you are actually multiplying the fractional number by 1 (the multiplicative identity), and you have not changed the value of the number.

11. If you divide both the numerator and denominator of the fraction $\frac{8}{16}$ by 8 you obtain $\frac{1}{2}$. Are $\frac{8}{16}$ and $\frac{1}{2}$ equivalent fractions?

12. In dividing both the numerator and denominator of $\frac{8}{16}$ by 8 you are actually dividing the fractional number by 1. Since dividing a number by 1 does not change its value, $\frac{1}{2}$ must be equivalent to $\frac{8}{16}$. Is $\frac{30}{36}$ equivalent to $\frac{5}{6}$? Why?

Rule: To find a fraction equivalent to a given fraction, we may divide the numerator and denominator by the same number.

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13. Use the rule to find the fraction equivalent to each of the following:

a. $\frac{3}{12}$ **b.** $\frac{6}{9}$ **c.** $\frac{9}{24}$ **d.** $\frac{18}{3}$ **e.** $\frac{15}{30}$ **f.** $\frac{12}{21}$

14. A fraction is in its *lowest terms* if the numerator and denominator cannot be divided by the same number. For example, $\frac{9}{16}$ is in lowest terms, since 3 is a factor of 9, but not a factor of 16. Also, 2 is a factor of 16, but not of 9.

When a fraction is in lowest terms, as $\frac{9}{16}$, the numerator and denominator are said to be *relatively prime*. Write in lowest terms the fraction equivalent to each of the following:

a. $\frac{6}{10}$ **c.** $\frac{8}{6}$ **e.** $\frac{21}{28}$ **g.** $\frac{21}{14}$ **i.** $\frac{9}{18}$

b. $\frac{7}{14}$ **d.** $\frac{16}{4}$ **f.** $\frac{15}{20}$ **h.** $\frac{18}{3}$ **j.** $\frac{4}{1000}$

15. Use the following property to determine if two fractions are equivalent:

Two fractions are equivalent if when renamed in lowest terms they are identical (have the same numerators and denominators).

Which of the following pairs are equivalent?

a. $\frac{3}{4}, \frac{12}{16}$ **c.** $\frac{4}{9}, \frac{36}{81}$ **e.** $\frac{2}{3}, \frac{30}{45}$

b. $\frac{5}{7}, \frac{20}{35}$ **d.** $\frac{11}{12}, \frac{110}{120}$ **f.** $\frac{5}{8}, \frac{35}{56}$

16. All members of a set of equivalent fractions name the same number. It is also true that, given a fraction, you can name many equivalent fractions. Thus any fractional number has many names. Name 5 fractions equivalent to $\frac{5}{8}$.

17. Name 5 fractions equivalent to $\frac{4}{7}$.

18. A fraction is said to be *in simplest form* when the numerator and denominator are relatively prime. Therefore, "in simplest form" and "in lowest terms" have the same meaning. We will hereafter use "in simplest form." Express each of the following in simplest form:

a. $\frac{4}{8}$ **b.** $\frac{3}{30}$ **c.** $\frac{100}{500}$ **d.** $\frac{27}{54}$ **e.** $\frac{13}{52}$ **f.** $\frac{9}{27}$

The following rule will simplify the procedure for expressing a fraction in simplest form:

Rule: To express fractions in simplest form, divide the numerator and denominator by their *greatest* common factor.

For example, 3 is a common factor of 9 and 27 in the fraction $\frac{9}{27}$ (or $\frac{27}{9}$), but 9 is the *greatest* common factor. Therefore, $\frac{9}{27}$ can be expressed in simplest form as $\frac{1}{3}$ by dividing the numerator and denominator by their greatest common factor, 9.

19. Consider writing $\frac{72}{84}$ in simplest form. Both 72 and 84 are divisible by 2. Why? Therefore, $\frac{72}{84} = \frac{36}{42}$. Divide both 36 and 42 by 2, and we have $\frac{18}{21}$. Now both 18 and 21 are divisible by 3, so an equivalent fraction is $\frac{6}{7}$. Since 6 and 7 are relatively prime, $\frac{72}{84}$ in simplest form is $\frac{6}{7}$. Notice that we divided the numerator and denominator by 2, by 2 again, and by 3. Would the result, $\frac{6}{7}$, be the same if we had divided 72 and 84 by $2 \times 2 \times 3$, or 12? Then 12 is the greatest common factor of 72 and 84.

Write each of the following in simplest form. (Your work will be simplified if you recognize the greatest common factor.)

a. $\frac{18}{24}$ **c.** $\frac{32}{40}$ **e.** $\frac{36}{63}$ **g.** $\frac{36}{54}$
b. $\frac{60}{96}$ **d.** $\frac{15}{90}$ **f.** $\frac{28}{70}$ **h.** $\frac{72}{192}$

20. There are times when the simplest form is not the most convenient. The machinist commonly expresses measurements in thousandths, even though they could be expressed in a simpler form. Express each of the following in thousandths:

a. $\frac{3}{100}$ **b.** $\frac{7}{10}$ **c.** $\frac{47}{100}$ **d.** $\frac{100}{10{,}000}$

21. Write each of the following as an equivalent fraction with the given denominator:

a. $\frac{3}{10} = \frac{?}{20}$ **c.** $\frac{3}{4} = \frac{?}{16}$ **e.** $\frac{4}{9} = \frac{?}{63}$
b. $\frac{4}{5} = \frac{?}{10}$ **d.** $\frac{70}{80} = \frac{?}{16}$ **f.** $\frac{7}{16} = \frac{?}{64}$

A *mixed numeral* is the indicated sum of a natural number and a fractional number. Thus $1\frac{2}{3}$ is actually $1 + \frac{2}{3}$. A number can be named by a mixed numeral.

Fractions that have the same denominator are called *like fractions*. To add numbers named by like fractions, add the numerators, and write the sum over the common denominator.

Thus: $$\frac{3}{3} + \frac{2}{3} = \frac{3+2}{3} = \frac{5}{3}$$

This is another way of saying that:

$$1\frac{2}{3} = \frac{5}{3}$$

EXAMPLE

What improper fraction is equivalent to $3\frac{4}{5}$?

By definition, $3\frac{4}{5} = 3 + \frac{4}{5}$

If $3 = \frac{5}{5} + \frac{5}{5} + \frac{5}{5}$

$= \frac{15}{5}$

then $3\frac{4}{5} = \frac{15}{5} + \frac{4}{5}$

and $3\frac{4}{5} = \frac{19}{5}$

1. Find the improper fraction equivalent to each of the following:

a. $4\frac{3}{4}$	**d.** $5\frac{2}{5}$	**g.** $3\frac{3}{7}$
b. $6\frac{3}{4}$	**e.** $2\frac{4}{9}$	**h.** $7\frac{5}{8}$
c. $6\frac{2}{3}$	**f.** $7\frac{1}{2}$	**i.** $2\frac{7}{16}$

To find a mixed numeral equivalent to an improper fraction, you could reverse the process in the above Example. It is quicker, however, to use the fact that a fraction is an expression of division.

EXAMPLE

Write $\frac{10}{3}$ as a mixed numeral.

$$\frac{10}{3} = 10 \div 3, \qquad 10 \div 3 = 3\frac{1}{3}$$

2. Use this method of finding the mixed numeral equivalent to each of the following:

a. $\frac{11}{3}$	**d.** $\frac{22}{7}$	**g.** $\frac{33}{10}$
b. $\frac{11}{8}$	**e.** $\frac{25}{12}$	**h.** $\frac{5}{2}$
c. $\frac{15}{4}$	**f.** $\frac{28}{8}$	**i.** $\frac{31}{16}$

To add fractional numbers named by like fractions:

1. Add the numerators;
2. Write the sum over the common denominator;
3. Write the answer in simplest form, renaming improper fractions as mixed numerals, and writing fractions in simplest form.

To add numbers named by mixed numerals:

1. Add the fractional numbers, writing the answer in simplest form;
2. Add the whole numbers.

These procedures are illustrated in the following Examples.

EXAMPLES

1. Add:

$$\begin{array}{r} \frac{1}{10} \\ \frac{7}{10} \\ \frac{3}{10} \\ \frac{9}{10} \\ \hline \frac{20}{10} \end{array} = 2$$

2. Add:

$$\begin{array}{r} 5\frac{5}{8} \\ 1\frac{3}{8} \\ 6\frac{7}{8} \\ 1\frac{5}{8} \\ \hline 13\frac{20}{8} \end{array}$$

Since $\frac{20}{8} = 2\frac{1}{2}$,

then $13\frac{20}{8} = 13 + 2\frac{1}{2} = 15\frac{1}{2}$

Add: Write all answers in simplest form.

1. $\begin{array}{r} \frac{3}{5} \\ \frac{1}{5} \\ \hline \end{array}$

2. $\begin{array}{r} \frac{3}{16} \\ \frac{7}{16} \\ \hline \end{array}$

3. $\begin{array}{r} 4\frac{1}{9} \\ 1\frac{5}{9} \\ \frac{7}{9} \\ \hline \end{array}$

4. $\begin{array}{r} 6\frac{3}{8} \\ 3\frac{5}{8} \\ \hline \end{array}$

5. $\begin{array}{r} 12\frac{1}{7} \\ 4\frac{3}{7} \\ \hline \end{array}$

6. $\begin{array}{r} 12\frac{9}{10} \\ \frac{3}{10} \\ 18\frac{7}{10} \\ \hline \end{array}$

7. $\begin{array}{r} 5\frac{7}{8} \\ 16\frac{5}{8} \\ \hline \end{array}$

8. $\begin{array}{r} 12\frac{1}{8} \\ 10\frac{5}{8} \\ \hline \end{array}$

9. $\begin{array}{r} \frac{1}{4} \\ 16\frac{1}{4} \\ 7\frac{3}{4} \\ \hline \end{array}$

10. $\begin{array}{r} 36\frac{1}{3} \\ 14\frac{1}{3} \\ \hline \end{array}$

11. $\begin{array}{r} 23\frac{5}{12} \\ 7\frac{7}{12} \\ \hline \end{array}$

12. $\begin{array}{r} 3\frac{2}{5} \\ 4\frac{4}{5} \\ 7\frac{3}{5} \\ \hline \end{array}$

13. Mr. Erickson is shipping 3 boxes. Their weight in pounds, respectively, is: $6\frac{1}{8}$, $7\frac{7}{8}$, and $19\frac{5}{8}$. What is their total weight in pounds?

14. Fred works at a filling station, and is paid at the rate of \$1.20 an hour. Last week he worked the following number of hours each day: Monday, $2\frac{3}{4}$; Tuesday, $3\frac{1}{4}$; Wednesday, 3; Thursday, $2\frac{1}{4}$; and Friday, $2\frac{3}{4}$. How much did he earn for the week?

To subtract fractional numbers named by like fractions, subtract the numerators, and write the difference over the common denominator. Write the answer in simplest form. In subtracting numbers named by mixed numerals:

1. Subtract the fractional numbers;
2. Subtract the whole numbers;
3. Write the answer in simplest form.

Example 2 below illustrates the use of these steps.

EXAMPLES

1. $\begin{array}{r} \frac{7}{9} \\ -\frac{4}{9} \\ \hline \frac{3}{9} = \frac{1}{3} \end{array}$

2. $\begin{array}{r} 8\frac{7}{8} \\ -2\frac{5}{8} \\ \hline 6\frac{2}{8} = 6\frac{1}{4} \end{array}$

You may need to regroup before subtracting the fractional numbers.

EXAMPLE

Subtract: $\begin{array}{r} 4\frac{3}{8} \\ -2\frac{5}{8} \\ \hline \end{array}$

We cannot subtract $\frac{5}{8}$ from $\frac{3}{8}$.

However, $4 = 3 + 1$
$= 3 + \frac{8}{8}$

Therefore, $4\frac{3}{8} = 3 + \frac{8}{8} + \frac{3}{8}$
$= 3 + \frac{11}{8}$

Thus $\begin{array}{r} 3\frac{11}{8} \\ -2\frac{5}{8} \\ \hline 1\frac{6}{8} = 1\frac{3}{4} \end{array}$

Subtract: Write all answers in simplest form.

1. $\begin{array}{r} 4\frac{9}{16} \\ -2\frac{3}{16} \\ \hline \end{array}$

2. $\begin{array}{r} 7\frac{5}{8} \\ -3\frac{3}{8} \\ \hline \end{array}$

3. $\begin{array}{r} 12\frac{9}{14} \\ -7\frac{11}{14} \\ \hline \end{array}$

4. $\begin{array}{r} 4\frac{7}{9} \\ -1\frac{5}{9} \\ \hline \end{array}$

5. $\begin{array}{r} 13\frac{11}{16} \\ -7\frac{13}{16} \\ \hline \end{array}$

6. $\begin{array}{r} 8\frac{1}{5} \\ -3\frac{4}{5} \\ \hline \end{array}$

7. $\begin{array}{r} 23\frac{1}{12} \\ -4\frac{5}{12} \\ \hline \end{array}$

8. $\begin{array}{r} 7\frac{3}{10} \\ -6\frac{9}{10} \\ \hline \end{array}$

9. The weight in pounds of a box of oranges is $42\frac{3}{8}$. The weight in pounds of the box alone is $2\frac{7}{8}$. What is the weight in pounds of the oranges, without the box?

10. The weight in pounds of four jars of maple syrup when packed for mailing is 7. The jars of syrup weigh, in pounds, $1\frac{1}{8}$ each, before packing. What is the weight of the packing material?

Fractions with different denominators are called *unlike fractions.* Since the denominators are not alike, we must express them as like fractions before we can add. Usually we express them by using the *least common denominator,* but it is not necessary to do this if a larger common denominator is more convenient. Remember! To be a common denominator, the number must be a multiple of each of the denominators.

EXAMPLES

1. Add: $\frac{1}{2} + \frac{5}{6} + \frac{7}{12} = ?$

In adding, always look to see if the greatest denominator is a common denominator. In this case it is. So we rewrite the other two fractions as equivalent fractions with denominator 12 and then add them.

$$\begin{array}{r} \frac{6}{12} \\ \frac{10}{12} \\ \underline{\frac{7}{12}} \\ \frac{23}{12} = 1\frac{11}{12} \end{array}$$

2. Add: $\frac{3}{8} + \frac{5}{12} + \frac{3}{4} = ?$

We cannot use 12 as the common denominator, as it is not a multiple of 8. However, 24 is a multiple of 12, 8, and 4; therefore, it is a common denominator. We rewrite the three fractions as equivalent fractions with denominator 24 and then add.

$$\begin{array}{r} \frac{9}{24} \\ \frac{10}{24} \\ \underline{\frac{18}{24}} \\ \frac{37}{24} = 1\frac{13}{24} \end{array}$$

Add:

1. $23\frac{7}{8}$ $18\frac{5}{6}$	**6.** $7\frac{1}{4}$ $4\frac{1}{3}$	**11.** $16\frac{2}{3}$ $24\frac{3}{4}$	**16.** $14\frac{1}{2}$ $7\frac{3}{4}$
2. $16\frac{3}{4}$ $19\frac{4}{5}$	**7.** $4\frac{1}{2}$ $2\frac{5}{7}$	**12.** $18\frac{1}{2}$ $9\frac{2}{5}$	**17.** $\frac{2}{3}$ $\frac{4}{5}$
3. $17\frac{5}{6}$ $38\frac{7}{8}$	**8.** $5\frac{8}{9}$ $\frac{3}{5}$	**13.** $12\frac{2}{3}$ $13\frac{1}{6}$	**18.** $\frac{5}{6}$ $\frac{5}{9}$
4. $\frac{1}{3}$ $\frac{5}{12}$ $\frac{5}{6}$	**9.** $\frac{5}{6}$ $\frac{2}{3}$ $\frac{3}{4}$	**14.** $\frac{1}{4}$ $\frac{3}{5}$ $\frac{3}{4}$	**19.** $\frac{3}{8}$ $\frac{5}{6}$ $\frac{3}{4}$
5. $\frac{7}{8}$ $\frac{1}{4}$ $\frac{1}{6}$	**10.** $\frac{9}{16}$ $\frac{5}{8}$ $\frac{1}{2}$	**15.** $\frac{5}{8}$ $\frac{5}{6}$ $\frac{2}{3}$	**20.** $\frac{15}{16}$ $\frac{3}{4}$ $\frac{5}{8}$

In finding the least common denominator we are required to find the *least common multiple,* L.C.M., of the denominators. There is a simple method for doing this. Using 8, 12, and 4, you can find the L.C.M. by using the following steps:

1. Write the prime factors of each denominator. See Example 2 on page 119.

   $8 = 2 \times 2 \times 2; \qquad 12 = 3 \times 2 \times 2; \qquad 4 = 2 \times 2$

2. Use each prime factor the greatest number of times it occurs in any one of the denominators as factors for the L.C.M.

The L.C.M. must contain all the prime factors of the denominators. In 8, 2 is used as a factor three times; in 12, 2 is used twice; in 4, 2 is used twice. Therefore, the L.C.M. must contain $2 \times 2 \times 2$ as factors. The only other factor is 3 which appears as a factor of 12. Thus

$$2 \times 2 \times 2 \times 3 = 24$$

is the L.C.M.

1. Find the L.C.M. of each of the following:

a. 8, 6	**d.** 2, 5	**g.** 16, 8, 2	**j.** 6, 9, 3
b. 4, 5	**e.** 2, 7	**h.** 3, 5, 6	**k.** 2, 3, 5
c. 3, 4	**f.** 9, 5	**i.** 8, 6, 4	**l.** 3, 7, 10

Notice that in exercises b, c, d, e, f, k, and l above, the numbers named are relatively prime. When this is the case, the L.C.M. is equal to the product of the given numbers. For example, in b, 4 and 5 are relatively prime. Therefore, the L.C.M. equals 4×5, or 20. In k, 2, 3, and 5 are relatively prime. Therefore, the L.C.M. equals $2 \times 3 \times 5$, or 30.

2. Using L.C M. find the sums of each of the following:

a. $2\frac{5}{12}$ $4\frac{3}{8}$	**d.** $31\frac{3}{5}$ $48\frac{4}{7}$	**g.** $63\frac{1}{6}$ $9\frac{7}{8}$	**j.** $95\frac{1}{2}$ $72\frac{3}{16}$
b. $\frac{1}{3}$ $\frac{3}{5}$ $\frac{7}{15}$	**e.** $\frac{3}{10}$ $\frac{5}{7}$ $\frac{1}{2}$	**h.** $\frac{1}{4}$ $\frac{5}{8}$ $\frac{6}{7}$	**k.** $\frac{8}{9}$ $\frac{2}{3}$ $\frac{1}{5}$
c. $3\frac{2}{3}$ $5\frac{5}{8}$	**f.** $21\frac{5}{7}$ $39\frac{2}{5}$	**i.** $37\frac{1}{2}$ $12\frac{2}{9}$	**l.** $15\frac{3}{4}$ $83\frac{2}{5}$

PROBLEMS: ADDITION OF FRACTIONAL NUMBERS

1. Harry rode his bicycle the following number of miles on Friday, Saturday, and Sunday respectively: $3\frac{1}{2}$, $5\frac{3}{10}$, $2\frac{9}{10}$. How many miles did he ride altogether?
2. On three days the snowfall in inches was: $1\frac{3}{10}$, $3\frac{7}{10}$, $4\frac{3}{10}$. What was the total snowfall for these three days?
3. Paul needs two rods. The measure of each rod in inches is $3\frac{5}{16}$ and $5\frac{13}{16}$. How much does he need altogether?
4. Ellen needs to cut a piece of cardboard into two strips, one measuring $\frac{5}{8}$ of an inch and the other $\frac{3}{4}$ of an inch in width. What should the original piece measure in width?
5. Jane needs $\frac{3}{4}$ of a yard of red broadcloth and $\frac{5}{6}$ of a yard of blue broadcloth. How much will it cost at $2.00 a yard?
6. The last four days, Mike practiced his violin the following number of hours: $\frac{3}{4}$, $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{1}{2}$. How many hours did he practice on all four days?
7. Mrs. Adams had $5\frac{1}{4}$ yards of curtain material. She bought an additional $\frac{7}{8}$ of a yard. How much material did she then have?
8. Mary had $\frac{1}{2}$ cup of syrup in one bottle and $\frac{2}{3}$ cup in another. How much did she have in all?
9. George glued together (side by side) four strips of wood whose measures in inches were: $3\frac{1}{2}$, $2\frac{5}{16}$, $3\frac{5}{8}$, and $4\frac{1}{4}$. What was the measure of the finished board?
10. Jane did baby-sitting for $1\frac{1}{2}$ hours on Friday and $\frac{3}{4}$ of an hour on Saturday. She received 60 cents an hour. How much did she earn?
11. Harry worked in the yard for Mrs. Ferris every Tuesday. On four Tuesdays last month he worked $1\frac{1}{2}$ hours, $2\frac{1}{4}$ hours, $1\frac{5}{6}$ hours, and $1\frac{3}{4}$ hours. How much did Harry earn at 75 cents per hour?
12. Mr. Craft spends $\frac{1}{4}$ of his salary for rent, $\frac{1}{3}$ for food, $\frac{1}{6}$ for clothing, and $\frac{1}{8}$ for auto expenses. The remainder of his salary is used for personal expenses and savings. What fraction of his salary is left for personal expenses and savings?
13. There are four brothers in the Baskin family. Their weights in pounds are: $106\frac{3}{8}$, $121\frac{3}{4}$, $129\frac{7}{8}$, $139\frac{1}{2}$ What is their total weight?
14. On a recent trip to Rochester, the Herringers spent $3\frac{3}{4}$ hours going, but only $2\frac{7}{8}$ hours on the return trip. What was the total travel time for the trip?
15. On the three days just prior to the Winter Carnival, it snowed the following number of inches: $2\frac{3}{4}$, $3\frac{1}{8}$, and $1\frac{5}{6}$. What was the total snowfall for the three days?

In subtraction problems that do not have a common denominator, we proceed as we did with addition problems.

1. Find the L.C.M.
2. Subtract the fractional numbers.
3. Subtract the whole numbers.

EXAMPLE

$$\begin{array}{rcr} 5\frac{3}{4} & = & 5\frac{9}{12} \\ -\ 2\frac{2}{3} & = & -\ 2\frac{8}{12} \\ & & 3\frac{1}{12} \end{array}$$

Subtract:

1. $\frac{7}{8} - \frac{1}{3}$
2. $\frac{5}{6} - \frac{1}{3}$
3. $1\frac{3}{4} - \frac{2}{3}$
4. $5\frac{7}{8} - 1\frac{5}{6}$
5. $\frac{1}{4} - \frac{3}{16}$
6. $11\frac{5}{6} - 5\frac{3}{4}$
7. $19\frac{3}{5} - 13\frac{1}{4}$
8. $\frac{4}{5} - \frac{2}{5}$
9. $\frac{5}{6} - \frac{3}{4}$
10. $4\frac{5}{8} - 2\frac{1}{2}$
11. $23\frac{9}{10} - 14\frac{1}{5}$
12. $6\frac{13}{16} - 3\frac{5}{8}$

EXAMPLE

$$\begin{array}{rcrcr} 8\frac{1}{2} & = & 8\frac{3}{6} & = & 7\frac{9}{6} \\ -\ 5\frac{2}{3} & = & -\ 5\frac{4}{6} & = & -\ 5\frac{4}{6} \\ & & & & 2\frac{5}{6} \end{array}$$

Look at the Example at the right. How was the common denominator found? Explain why $8\frac{3}{6} = 7\frac{9}{6}$.

13. $5\frac{1}{4} - 2\frac{3}{4}$
14. $9\frac{3}{4} - 5\frac{4}{5}$
15. $22\frac{1}{2} - 16\frac{2}{3}$
16. $16\frac{3}{8} - 14\frac{3}{4}$
17. $7\frac{3}{5} - 3\frac{4}{5}$
18. $6\frac{3}{8} - 4\frac{3}{4}$
19. $30\frac{3}{5} - 21\frac{4}{15}$
20. $16\frac{1}{8} - 12\frac{1}{3}$
21. $8\frac{3}{8} - 6\frac{3}{4}$
22. $13\frac{1}{12} - 7\frac{5}{8}$
23. $9 - 5\frac{2}{3}$
24. $14\frac{4}{5} - 13\frac{1}{2}$
25. $17 - 4\frac{3}{7}$
26. $20\frac{1}{3} - 8\frac{5}{6}$
27. $28\frac{3}{10} - 21\frac{2}{5}$
28. $6\frac{5}{12} - 5\frac{2}{3}$
29. $19\frac{1}{2} - 12\frac{5}{7}$
30. $8\frac{1}{4} - 5\frac{1}{3}$
31. $14\frac{5}{8} - 11\frac{11}{16}$
32. $16\frac{1}{2} - 9\frac{7}{10}$
33. $8\frac{3}{7} - 4\frac{3}{14}$
34. $32\frac{1}{8} - 9\frac{5}{12}$
35. $16 - 15\frac{13}{15}$
36. $23\frac{1}{5} - 13\frac{1}{3}$

1. How much longer is a ribbon that measures $\frac{5}{6}$ of a yard than a ribbon that measures $\frac{1}{3}$ of a yard?
2. Ellen finds she has used only $\frac{1}{4}$ of a teaspoon of lemon extract in a recipe. She needs $\frac{2}{3}$ of a teaspoon. How much more must she add?
3. How much greater is the thickness of a board that measures $1\frac{1}{8}''$ than one that measures $\frac{3}{4}''$ in thickness?
4. George had $\frac{3}{4}$ of a pound of candy to sell. After he had sold $\frac{1}{4}$ of a pound, how much did he have left?
5. Rose has $\frac{1}{2}$ cup of syrup. The recipe calls for $\frac{2}{3}$ cup. How much more does she need?
6. A camera, when packed for overseas shipment, weighs $4\frac{1}{4}$ pounds. When packed for shipment in this country it weighs only $3\frac{7}{8}$ pounds. How much does the extra packing for overseas shipment weigh?
7. A large tube of tooth paste weighs $6\frac{1}{8}$ ounces. A smaller tube weighs $3\frac{1}{2}$ ounces. What is the difference in weight?
8. A 9-inning baseball game was halted by rain after $6\frac{2}{3}$ innings. How many innings remain to be played when play is resumed?
9. What is the difference in thickness between a board that measures $3\frac{3}{4}$ inches and one that measures $5\frac{3}{8}$ inches?
10. How much heavier is a $67\frac{3}{8}$-pound package than a $66\frac{3}{4}$-pound package?
11. Sam rode his bicycle $8\frac{7}{10}$ miles and Henry rode his $9\frac{1}{10}$ miles. How much farther did Henry ride than Sam?
12. How much more than $\frac{3}{4}$ of a cup is $1\frac{1}{6}$ cups?
13. George is delivering material to a city $121\frac{3}{4}$ miles away. After driving $73\frac{2}{10}$ miles he stopped for lunch. How much farther does he have to drive?
14. The gasoline tank in the Jensen's car holds 17 gallons. After a recent trip it took $12\frac{3}{10}$ gallons to fill it. How many gallons were in the tank before it was filled?
15. Fred cut a piece of board that measures $8\frac{3}{4}''$ in length from a board that measures $18''$ in length. What length of board was left?
16. A typewriter weighs $26\frac{3}{4}$ pounds. When packed for shipping, it weighs $41\frac{1}{8}$ pounds. How much do the packing materials weigh?
17. Bill weighs $121\frac{3}{8}$ pounds, and his brother weighs $115\frac{3}{4}$ pounds. How much heavier is Bill than his brother?
18. Six months ago Ken was 5 feet $2\frac{3}{4}$ inches tall. Now he is 5 feet $3\frac{1}{8}$ inches tall. How much has he grown during the past six months?

For each statement, first write the equivalent statement that has the variable alone on one side. Then solve and check by substituting the value for the variable in the original statement.

EXAMPLES

1. $17\frac{2}{3} + N = 43\frac{1}{3}$ $\qquad a + b = s$
   $N = 43\frac{1}{3} - 17\frac{2}{3}$ $\qquad b = s - a$
   $N = 25\frac{2}{3}$
   Check: $17\frac{2}{3} + 25\frac{2}{3} = 43\frac{1}{3}$

2. $56\frac{1}{4} - N = 27\frac{3}{4}$ $\qquad s - a = b$
   $N = 56\frac{1}{4} - 27\frac{3}{4}$ $\qquad a = s - b$
   $N = 28\frac{1}{2}$
   Check: $56\frac{1}{4} - 28\frac{1}{2} = 27\frac{3}{4}$

1. $N + 5\frac{1}{2} = 25$
2. $N = 13 + 27\frac{1}{2}$
3. $24 - N = 16\frac{3}{4}$
4. $39 - N = 25$
5. $35\frac{1}{2} = N + 18\frac{1}{3}$
6. $N - 28\frac{1}{2} = 41\frac{1}{4}$
7. $56\frac{1}{4} - N = 21\frac{1}{3}$
8. $19\frac{5}{8} + N = 33\frac{1}{2}$
9. $N - 16\frac{2}{3} = 14\frac{1}{2}$
10. $N = 35\frac{7}{8} - 16\frac{2}{3}$
11. $N + 18\frac{1}{3} = 35\frac{1}{2}$
12. $N + 15\frac{3}{8} = 33\frac{1}{3}$
13. $19\frac{3}{4} - 13\frac{5}{8} = N$
14. $N - 11\frac{1}{2} = 13\frac{1}{3}$
15. $37\frac{1}{4} - N = 23\frac{5}{6}$
16. $19\frac{3}{7} - N = 11\frac{9}{14}$
17. $N + 17\frac{3}{8} = 25\frac{1}{2}$
18. $58\frac{1}{5} + N = 62\frac{1}{2}$
19. $N - 29\frac{5}{9} = 11\frac{1}{3}$
20. $28\frac{2}{3} - N = 3\frac{1}{4}$
21. $N - 17\frac{1}{4} = 22\frac{1}{2}$
22. $27\frac{1}{4} - 19 = N$
23. $18 + N = 48$
24. $N = 28\frac{1}{4} + 13\frac{2}{3}$
25. $N - 14 = 3\frac{1}{5}$
26. $N + 36 = 53\frac{1}{2}$
27. $17\frac{3}{4} + N = 39\frac{7}{8}$
28. $N = 13\frac{2}{3} + 57\frac{1}{4}$
29. $37\frac{1}{2} + N = 83\frac{1}{3}$
30. $33\frac{1}{2} + 15\frac{1}{8} + 25\frac{3}{5} = N$
31. $37\frac{3}{4} - N = 18\frac{2}{3}$
32. $37\frac{3}{8} = 13\frac{1}{4} + N$
33. $N = 11\frac{2}{3} + 15\frac{7}{8}$
34. $N + 56\frac{1}{4} = 62\frac{1}{2}$
35. $18\frac{9}{10} + N = 32\frac{2}{3}$
36. $N = 17\frac{1}{7} - 9\frac{1}{3}$
37. $57\frac{2}{3} - N = 18\frac{5}{9}$
38. $38\frac{2}{3} - 18\frac{3}{4} = N$
39. $26\frac{1}{6} + N = 54\frac{3}{4}$
40. $81\frac{1}{9} = N + 45\frac{1}{3}$
41. $14\frac{1}{8} = N - 16\frac{1}{4}$
42. $N + 8\frac{3}{5} = 36\frac{7}{10}$
43. $27\frac{2}{7} - 8\frac{9}{14} = N$
44. $N = 5\frac{1}{6} + 13\frac{7}{8}$

A. Express each of the following in simplest form.

1. $\frac{6}{8}$
2. $\frac{5}{15}$
3. $\frac{28}{32}$
4. $\frac{15}{25}$
5. $\frac{8}{12}$
6. $\frac{9}{15}$
7. $\frac{12}{16}$
8. $\frac{27}{33}$
9. $\frac{14}{21}$
10. $\frac{9}{12}$
11. $\frac{6}{18}$
12. $\frac{8}{16}$
13. $\frac{15}{36}$
14. $\frac{26}{39}$
15. $\frac{36}{45}$
16. $\frac{32}{40}$

B. Find the least common denominator for each group of fractions.

1. $\frac{1}{8}, \frac{3}{4}, \frac{1}{2}$
2. $\frac{5}{6}, \frac{2}{3}, \frac{5}{12}$
3. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}$
4. $\frac{5}{8}, \frac{3}{4}, \frac{5}{6}$
5. $\frac{4}{5}, \frac{2}{3}, \frac{3}{4}$
6. $\frac{7}{10}, \frac{1}{4}, \frac{3}{5}$
7. $\frac{3}{8}, \frac{1}{4}, \frac{1}{3}$
8. $\frac{1}{6}, \frac{4}{5}, \frac{2}{3}$
9. $\frac{2}{9}, \frac{5}{6}, \frac{1}{2}$
10. $\frac{3}{4}, \frac{2}{5}, \frac{1}{3}$
11. $\frac{2}{3}, \frac{1}{6}, \frac{5}{7}$
12. $\frac{5}{12}, \frac{2}{3}, \frac{3}{4}$
13. $\frac{1}{8}, \frac{7}{12}, \frac{1}{3}$
14. $\frac{5}{18}, \frac{5}{6}, \frac{1}{12}$
15. $\frac{2}{5}, \frac{4}{7}, \frac{1}{2}$
16. $\frac{1}{13}, \frac{2}{3}, \frac{1}{6}$
17. $\frac{3}{8}, \frac{2}{5}, \frac{1}{10}$
18. $\frac{3}{10}, \frac{1}{15}, \frac{1}{6}$

C. *Add:* Write all answers in simplest form.

1. $\frac{1}{2} + \frac{5}{6}$
2. $\frac{5}{16} + \frac{5}{8}$
3. $13\frac{3}{5} + 8\frac{9}{10} + 5\frac{1}{2}$
4. $\frac{1}{8} + \frac{1}{3}$
5. $\frac{4}{5} + \frac{3}{4} + \frac{1}{2}$
6. $16\frac{5}{8} + 4\frac{1}{6}$
7. $\frac{2}{5} + 3\frac{1}{2} + 7$
8. $\frac{5}{12} + \frac{5}{6}$
9. $8\frac{2}{3} + 4\frac{7}{12} + 18\frac{1}{4}$
10. $15\frac{5}{6} + 14\frac{1}{2} + 7\frac{2}{3} + 8\frac{1}{2}$
11. $\frac{5}{8} + \frac{5}{12}$
12. $\frac{1}{2} + 3\frac{1}{4} + \frac{5}{8}$
13. $14\frac{3}{5} + 24\frac{2}{7}$
14. $\frac{3}{8} + \frac{5}{6} + 3\frac{1}{3} + 7\frac{3}{4}$
15. $12\frac{1}{13} + 18\frac{1}{12}$
16. $3\frac{5}{18} + 2\frac{1}{6} + 5\frac{7}{12}$
17. $\frac{2}{3} + \frac{1}{6} + \frac{3}{7}$
18. $\frac{3}{10} + \frac{2}{15} + \frac{5}{6}$

D. *Subtract:* Write all answers in simplest form.

1. $\frac{7}{8} - \frac{3}{4}$
2. $12\frac{7}{12} - 3\frac{1}{4}$
3. $6\frac{5}{8} - 4\frac{3}{4}$
4. $13 - 12\frac{5}{6}$
5. $14\frac{2}{3} - 13\frac{4}{5}$
6. $5\frac{5}{6} - 2\frac{1}{3}$
7. $13\frac{9}{16} - 7\frac{3}{8}$
8. $8\frac{1}{2} - 1\frac{2}{3}$
9. $9\frac{5}{6} - 2\frac{7}{8}$
10. $7\frac{3}{8} - 5\frac{2}{5}$
11. $9\frac{5}{7} - 6\frac{1}{2}$
12. $12\frac{13}{15} - 8\frac{4}{5}$
13. $7 - 5\frac{4}{9}$
14. $8\frac{3}{5} - 6\frac{3}{4}$
15. $13\frac{3}{10} - 9\frac{1}{15}$
16. $21\frac{3}{8} - 7\frac{4}{5}$
17. $17\frac{1}{9} - 6\frac{2}{3}$
18. $18 - 12\frac{3}{5}$

If you need more practice, turn to the Practice Exercises on page 445 and following. If not, you may work in the Experts' Corner on the following page.

A Magic Square

See if you can work these problems without using a pencil. Write down only the answer as a fraction in simplest form.

Draw a square like the one at the right, only without the numerals in parentheses. Write the answers to problems 1, 2, and 3 in the top row; 4, 5, and 6 in the second row; and 7, 8, and 9 in the bottom row. When you are finished, you should have a magic square.

(1)	(2)	(3)
(4)	(5)	(6)
(7)	(8)	(9)

1. John earned \$15 last week and Henry earned \$3. What was the ratio of Henry's earnings to John's?

2. The basketball team played 20 games and won 18 of them. What part of its games did the team win?

3. Ethel has an allowance of \$5.50 a week. She spends \$2.20 a week for lunches. What part of her allowance is this?

4. Mr. Jones was elected mayor of the village at the last election. Of 2000 votes cast he received 1400. What part of the votes did he receive?

5. In a recent football game Roger Brown, the quarterback, completed 13 of 26 passes. What part of his passes was completed?

6. During physical education classes last week, Eric tried 50 shots from the floor, and sank 15 of them. What part of his shots were baskets?

7. There are 4000 registered voters in Rosedale. At a recent election 2400 of them voted. What fraction of them voted?

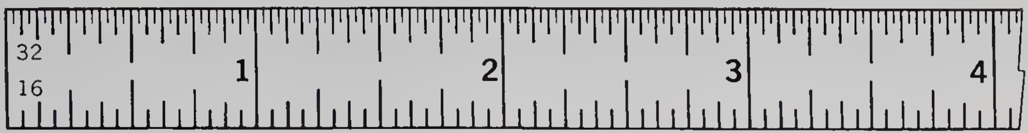
8. There are 40 pupils in Miss Brown's speech class. Yesterday 36 pupils were present. What part of the class was absent?

9. Of 1500 homes in Roseville, 1200 have refrigerators. What fraction of the homes have refrigerators?

10. Now test your square to see if it is magic. Add across each row, down each column, and from corner to corner for each diagonal. If they do not all add up to the same total, you have made a mistake.

11. Make another magic square, with nine squares. Then prepare nine problems whose answers are the numbers named in the squares. Replace the answers in the squares with numerals in parentheses, like those in the Figure you started with above. See how many in your class can solve your magic square the first time.

1. How many inches is $4 \times \frac{1}{2}''$? Count off four half-inch spaces on your ruler, and see if you are right.



2. Use the ruler to help you find the answers to these exercises.

a. $3 \times \frac{1}{4}$
b. $4 \times \frac{3}{4}$
c. $2 \times \frac{3}{8}$
d. $\frac{1}{4}$ of 3
e. $\frac{3}{4}$ of 4
f. $\frac{3}{8}$ of 2
g. $\frac{1}{2} \times \frac{1}{2}$
h. $\frac{1}{3} \times \frac{3}{4}$
i. $\frac{1}{4} \times 4$
j. $\frac{1}{5} \times \frac{5}{16}$
k. $\frac{1}{2} \times \frac{1}{4}$
l. $\frac{1}{4} \times \frac{1}{2}$
m. $\frac{1}{16} \times 2$
n. $\frac{1}{3} \times \frac{3}{8}$
o. $\frac{1}{6} \times 3$
p. $\frac{1}{2} \times \frac{1}{16}$
q. $\frac{2}{3} \times \frac{3}{4}$
r. $\frac{5}{8} \times 4$

Have you discovered the rule for multiplying fractional numbers?

Rule: The product of two fractional numbers is a fractional number such that the numerator of the new fraction is the product of the numerators, and the denominator of the new fraction is the product of the denominators.

In using this rule you should name the numbers as improper fractions, as shown in Examples 2 and 3.

EXAMPLES

1. $\frac{3}{5} \times \frac{7}{8} = ?$

$$\frac{3 \times 7}{5 \times 8} = \frac{21}{40}$$

2. $2\frac{1}{4} \times 7\frac{1}{3} = ?$

$$\frac{9}{4} \times \frac{22}{3} = \frac{198}{12} = 16\frac{1}{2}$$

3. $9 \times \frac{5}{12} = ?$

$$\frac{9}{1} \times \frac{5}{12} = \frac{45}{12} = 3\frac{3}{4}$$

You remember that a number divided by 1 equals the number. Thus $\frac{9}{1}$ and 9 are equivalent (Example 3 above).

3. Use the rule to find the following products:

a. $\frac{3}{4} \times \frac{16}{25}$
b. $\frac{5}{8} \times 24$
c. $9 \times \frac{5}{6}$
d. $3\frac{1}{2} \times \frac{3}{4}$
e. $4\frac{2}{3} \times \frac{3}{4}$
f. $1\frac{1}{2} \times 2\frac{3}{4}$
g. $50 \times \frac{3}{5}$
h. $\frac{5}{6} \times 9$
i. $1\frac{2}{3} \times \frac{4}{15}$
j. $6\frac{1}{2} \times 4\frac{2}{5}$
k. $3\frac{1}{3} \times 1\frac{9}{10}$
l. $2\frac{1}{2} \times 3\frac{2}{3}$
m. $5\frac{1}{3} \times 6\frac{3}{5}$
n. $2\frac{7}{8} \times \frac{4}{7}$
o. $36 \times 1\frac{5}{6}$
p. $2\frac{5}{9} \times 2\frac{1}{4}$
q. $\frac{3}{4} \times \frac{9}{16}$
r. $2\frac{1}{2} \times \frac{2}{5}$

Most problems involving multiplication with fractions can be simplified by using what you know about multiplication.

EXAMPLE

Find: $1\frac{5}{9} \times 2\frac{1}{7} = ?$

a. Rewrite as improper fractions: $\frac{14}{9} \times \frac{15}{7}$

b. Rewrite in prime factors: $\frac{2 \times 7 \times 5 \times 3}{3 \times 3 \times 7}$

c. Rearrange and regroup the factors, using the commutative and associative properties of multiplication: $\frac{3 \times 7 \times 2 \times 5}{3 \times 7 \times 3}$

$\frac{3 \times 7}{3 \times 7} \times \frac{2 \times 5}{3}$

d. $\frac{3 \times 7}{3 \times 7} = 1$

Since 1 is the multiplicative identity, this can be rewritten as: $1 \times \frac{10}{3} = \frac{10}{3}$
$= 3\frac{1}{3}$

Let's try another example. Explain each step, as is done on the left in the above Example.

EXAMPLE

Find: $2\frac{7}{10} \times 6\frac{2}{3} = ?$

a. $2\frac{7}{10} \times 6\frac{2}{3} = \frac{27}{10} \times \frac{20}{3}$

b. $\frac{3 \times 3 \times 3 \times 2 \times 2 \times 5}{2 \times 5 \times 3} =$

c. $\frac{2 \times 5 \times 3 \times 3 \times 3 \times 2}{2 \times 5 \times 3} =$

d. $\frac{2 \times 5 \times 3}{2 \times 5 \times 3} \times \frac{3 \times 3 \times 2}{1} =$

e. $1 \times \frac{18}{1} = 18$

Use the above procedure to find these products:

1. $\frac{2}{3} \times 9$
2. $8\frac{1}{3} \times \frac{3}{5}$
3. $\frac{5}{8} \times \frac{12}{25}$
4. $1\frac{5}{9} \times \frac{5}{7}$
5. $36 \times 6\frac{1}{4}$
6. $3\frac{3}{4} \times \frac{4}{15}$
7. $12 \times \frac{3}{4}$
8. $6\frac{2}{3} \times \frac{12}{25}$
9. $7\frac{1}{2} \times \frac{4}{5}$
10. $1\frac{4}{9} \times \frac{3}{13}$
11. $21 \times \frac{13}{14}$
12. $\frac{5}{8} \times \frac{8}{5}$
13. $5\frac{5}{6} \times \frac{6}{7}$
14. $6\frac{2}{5} \times \frac{5}{8}$
15. $2\frac{4}{5} \times \frac{10}{21}$
16. $\frac{5}{11} \times 3\frac{1}{7}$
17. $2\frac{8}{9} \times 13\frac{1}{2}$
18. $3\frac{1}{4} \times 1\frac{7}{13}$

Usually you can recognize the *common* factor or factors in the numerator and denominator, if there are any, without using all the steps shown in Examples on page 128. Examine the following Example and see how some of the steps are bypassed.

EXAMPLE

Find: $\frac{2}{3} \times \frac{15}{16}$

a. $\frac{2}{3} \times \frac{15}{16} = \frac{2 \times 15}{3 \times 16}$

b. $= \frac{1 \times 15}{3 \times 8}$ Dividing numerator and denominator by 2. Why?

c. $= \frac{1 \times 5}{1 \times 8} = \frac{5}{8}$ Dividing numerator and denominator by 3. Why?

All of this can be simplified to one step:

$$\frac{\overset{1}{\cancel{2}}}{\underset{1}{\cancel{3}}} \times \frac{\overset{5}{\cancel{15}}}{\underset{8}{\cancel{16}}} = \frac{1 \times 5}{1 \times 8} = \frac{5}{8}$$

Explain what the crossing out means, and what it is for.

Use the procedure illustrated above in the exercises that follow. Sometimes you will find an alternative method to be simpler in multiplying a whole number by a fractional number. This is illustrated in the two procedures in this Example.

EXAMPLE

Find: $2\frac{5}{6} \times 36$

(1) $\frac{17}{\underset{1}{\cancel{6}}} \times \overset{6}{\cancel{36}} = 102$

(2)
$$\begin{array}{r} 36 \\ 2\frac{5}{6} \\ \hline 30 \\ 72 \\ \hline 102 \end{array}$$

Explain what was done in (2).

In the following exercises you may use either form (1) or (2).

19. $34 \times 12\frac{1}{2}$

20. $24 \times 32\frac{3}{4}$

21. $35 \times 15\frac{2}{7}$

22. $18 \times 26\frac{2}{3}$

23. $72 \times 15\frac{3}{8}$

24. $42 \times 26\frac{5}{7}$

25. $39 \times 18\frac{3}{4}$

26. $42 \times 29\frac{3}{5}$

27. $21\frac{3}{8} \times 18\frac{3}{4}$

28. $19\frac{3}{4} \times 45\frac{1}{2}$

29. $16\frac{1}{6} \times 42\frac{3}{4}$

30. $11\frac{1}{5} \times \frac{5}{56}$

In general, fractional numbers are used in problems for three major purposes. These may be described as follows:

EXAMPLES

(a) *Dividing the whole of anything into parts:*

The senior class collected $\frac{2}{3}$ of the money necessary for the graduation dance at a school carnival and $\frac{1}{5}$ of the money with a candy sale. What fraction remained to be collected?

Solution:

$$\frac{2}{3} + \frac{1}{5} + ? = 1$$
$$\frac{2}{3} + \frac{1}{5} = \frac{10}{15} + \frac{3}{15} = \frac{13}{15}$$
$$1 - \frac{13}{15} = \frac{15}{15} - \frac{13}{15} = \frac{2}{15}$$

Answer: $\frac{2}{15}$

(b) *Taking a portion of a whole:*

Of the sophomore class at Clarkton High School, $\frac{5}{6}$ were born in Clarkton. If there were 282 students in the sophomore class, how many were born in Clarkton?

Solution:

$$\frac{5}{\cancel{6}_{1}} \times \cancel{282}^{47} = 235$$

Answer: 235

(c) *To compare two quantities:*

Mr. Edmonds' physics class of 28 students contained 16 boys. What fraction of the class was boys?

Solution: $\frac{16}{28} = \frac{4}{7}$

Answer: $\frac{4}{7}$

Classify each of the problems on the next page as (a), (b), or (c) according to the three Examples just discussed. Then complete the solution, using pencil and paper for your computations as little as possible. Be sure to follow the problem-solving steps in analyzing the problems.

STEPS FOR SOLVING APPLIED PROBLEMS

1. Understand the problem.	2. Note what the problem asks for.	3. Look for hidden questions.
6. Check your answer.	5. Set up and solve the conditional statement(s).	4. Estimate a reasonable answer.

1. A survey showed that of 600 pupils in Hawthorne High School, $\frac{3}{4}$ participated in some school activity. How many participated in a school activity?
2. Out of a graduating class of 250 at Hawthorne High School, 150 planned to go on to college. What fraction planned to go to college?
3. The Hawthorne High School football team won 8 games and lost two games this year. The number of games won was what fraction of the total number of games played?
4. Harry, Lloyd, and Bill bought a boat. Harry paid $\frac{1}{3}$ of the cost, Lloyd $\frac{1}{2}$ of the cost, and Bill the rest. What fraction of the cost did Bill pay?
5. The boat cost \$120. How much did each of the boys pay?
6. Helen went on a trip of 400 miles, $\frac{3}{4}$ of the way on the train and the rest by bus. How many miles did she travel in each way?
7. Fred was absent only 3 days out of the total school term of 180 days. During what fraction of the total school term was Fred absent?
8. Mike mowed $\frac{1}{2}$ of the lawn, and Carl mowed $\frac{1}{4}$. How much was left for Jim to mow, if he finished it?
9. Eric earned \$420 last summer. He saved \$160 of it for his college fund. What part of his earnings did he save?
10. Jane traveled 150 miles by bus and automobile. She traveled $\frac{2}{3}$ of the way by bus. How far did she travel by automobile?
11. One-fifth of Mr. Reed's chemistry class received the top grade on a recent test. There are 30 pupils in the class. How many did not receive the top grade?
12. Mary had \$2. Lorraine has $\frac{1}{4}$ as much money as Mary. How much does Lorraine have?
13. What fraction of an hour is 32 minutes?
14. Mr. Allen plans to plant $\frac{3}{4}$ of his 160 acre farm in corn, $\frac{1}{8}$ in potatoes, and the rest in tomatoes. How many acres of tomatoes does he plan to plant?
15. Yankee Stadium in New York has a seating capacity of 67,000 persons. If 44,000 attended a game, what fraction of the stadium was not filled?
16. Helen kept track of the amount of time she spends in various ways. She reports that she divides her time in this way:

Sleep: 8 hours	Meals: 2 hours
School: 7 hours	Homework: 3 hours
Recreation: the remainder	

What fraction of each 24-hour day does she spend in each way?

The Multiplicative Inverse

1. Find the products:

a. $\frac{2}{3} \times \frac{3}{2}$ **b.** $\frac{4}{7} \times \frac{7}{4}$ **c.** $\frac{5}{9} \times 1\frac{4}{5}$ **d.** $\frac{1}{4} \times 4$

2. If the product of two numbers is 1, then each number is the *multiplicative inverse* of the other. From Exercise 1 you can see that $\frac{2}{3}$ is the multiplicative inverse of $\frac{3}{2}$. Also $\frac{3}{2}$ is the multiplicative inverse of $\frac{2}{3}$. Another name for multiplicative inverse is *reciprocal.*

What is the multiplicative inverse of each of the following?

a. $\frac{4}{7}$ **b.** $\frac{7}{4}$ **c.** $1\frac{4}{5}$ **d.** 4 **e.** $\frac{1}{4}$ **f.** $\frac{6}{5}$

3. Find the multiplicative inverse of 8.

If $8n = 1$,	If $x \times y = p$,	
then $n = 1 \div 8$	then $y = p \div x$	
$1 \div 8 = \frac{1}{8}$, by definition of a fraction.		
Also, $8 = 1 \div n$	$x = p \div y$	

Then the multiplicative inverse of 8 is $\frac{1}{8}$. This tells us that:

---

(1) 1 divided by a non-zero number gives the multiplicative inverse of the number;

(2) 1 divided by the multiplicative inverse of the number gives the number.

Find the multiplicative inverse of each of the following. Check by showing that the product of the number and its multiplicative inverse is 1.

a. 7 **b.** 16 **c.** 25 **d.** 17 **e.** $\frac{8}{9}$ **f.** $\frac{5}{8}$ **g.** $\frac{16}{5}$

4. Find the value for n that makes each of these a true statement.

a. $n \times \frac{3}{4} = 1$
b. $\frac{5}{6} \times n = 1$
c. $3\frac{1}{8} \times \frac{8}{25} = n$
d. $1 \times n = 1$
e. $1\frac{2}{5} \times n = 1$
f. $\frac{1}{9} \times n = 1$
g. $n \times 11 = 1$
h. $2\frac{1}{2} \times n = 1$
i. $6\frac{2}{3} \times \frac{3}{20} = n$

5. Find the multiplicative inverse of $7\frac{3}{4}$.
HINT: First write $7\frac{3}{4}$ as an improper fraction.

6. Write the multiplicative inverse for each of the following:

a. $\frac{3}{8}$ **b.** $1\frac{4}{5}$ **c.** $\frac{1}{7}$ **d.** $4\frac{1}{2}$ **e.** $\frac{9}{16}$

7. Is this a true statement? $16 \div 4 = 16 \times \frac{1}{4}$

8. Let's examine the following statement:

The operation of division may be replaced by an equivalent multiplication, using the multiplicative inverse of the divisor as a factor.

With each of the following, see if multiplying by the multiplicative inverse of the divisor gives the same result as the indicated division.

a. $25 \div 5$ **b.** $48 \div 6$ **c.** $27 \div 3$ **d.** $18 \div 9$

9. Why should the statement, $25 \div 5 = 25 \times \frac{1}{5}$, be true?
In dividing $25 \div 5$ we are asking: How many 5's are in 25?
First, how many 5's are in 1? That is, $1 \div 5 = ?$
We know that $1 \div 5 = \frac{1}{5}$, which is the multiplicative inverse of 5.
There are 25 times as many 5's in 25 as there are in 1.
Then $25 \div 5 = 25 \times \frac{1}{5}$.
Explain in the same way why $42 \div 7 = 42 \times \frac{1}{7}$.

10. Does multiplying by n give the same result as dividing by $\frac{1}{n}$?

Does $7 \times 4 = 7 \div \frac{1}{4}$?

a. How many $\frac{1}{4}$'s are in 1? This is the same as asking, what is the multiplicative inverse of $\frac{1}{4}$? The answer is 4.
b. The number of $\frac{1}{4}$'s in 7 is how many times the number of $\frac{1}{4}$'s in 1?
c. What is the value of each side of the equation? $7 \div \frac{1}{4} = 7 \times 4$
d. Show in the same way that: $9 \div \frac{1}{5} = 9 \times 5$

11. Is this a true statement? $8 \div \frac{4}{5} = 8 \times \frac{5}{4}$

a. How many $\frac{4}{5}$'s are in 1?
b. The number of $\frac{4}{5}$'s in 8 is how many times the number in 1?
c. Find the value of both sides of the equation above.
d. Show in the same way that: $5 \div \frac{2}{3} = 5 \times \frac{3}{2}$

12. Does the rule hold in dividing one non-zero fractional number by another? Consider the following Example:

EXAMPLE

Divide: $\frac{5}{6} \div \frac{2}{3} = ?$

a. How many $\frac{2}{3}$'s are in 1?
b. Are there more or less than that in $\frac{5}{6}$?
c. Complete the Example.

The number of $\frac{2}{3}$'s in 1 is $\frac{3}{2}$.
The number of $\frac{2}{3}$'s in $\frac{5}{6}$ is $\frac{5}{6}$ times the answer to a, $\frac{3}{2}$.
Then: $\frac{5}{6} \div \frac{2}{3} = \frac{5}{6} \times \frac{3}{2}$

Show in the same way that

$$\begin{aligned}\tfrac{3}{8} \div \tfrac{5}{16} &= \tfrac{3}{8} \times (1 \div \tfrac{5}{16})\\ &= \tfrac{3}{8} \times \tfrac{16}{5}\end{aligned}$$

13. Does the rule hold for division of two numbers expressed as mixed numerals? Consider the following Example.

EXAMPLE

Divide: $6\frac{2}{3} \div 2\frac{1}{2} = ?$

a. The number of $2\frac{1}{2}$'s in 1 is $\frac{2}{5}$. How do you know that this is true?

b. Are there more or less than that in $6\frac{2}{3}$? The number of $2\frac{1}{2}$'s in $6\frac{2}{3}$ is $\frac{20}{3}$ as great.

c. Complete the Example. Then: $6\frac{2}{3} \div 2\frac{1}{2} = \frac{20}{3} \times (1 \div \frac{5}{2})$
$= \frac{20}{3} \times \frac{2}{5}$

Show in the same way that

$$\begin{aligned}5\tfrac{1}{2} \div 1\tfrac{2}{3} &= 5\tfrac{1}{2} \times (1 \div 1\tfrac{2}{3})\\ &= 5\tfrac{1}{2} \times \tfrac{3}{5}\end{aligned}$$

14. Illustrate by an example the use of the multiplicative inverse in dividing a number expressed as a mixed numeral by a fractional number. Explain why each step is true.

15. If n is any non-zero fractional number, are these true statements?

$$\begin{aligned}6 \div n &= 6 \times (1 \div n)\\ &= 6 \times \frac{1}{n}\end{aligned}$$

Explain why each step is true.

16. If both a and n are any non-zero fractional numbers, are these statements true?

$$\begin{aligned}a \div n &= a \times (1 \div n)\\ &= a \times \frac{1}{n}\end{aligned}$$

Explain why each step is true.

17. Use the rules you developed above to find the following quotients:

a. $\frac{3}{4} \div \frac{9}{10}$
b. $\frac{7}{8} \div \frac{15}{16}$
c. $\frac{5}{9} \div \frac{1}{6}$
d. $\frac{5}{6} \div \frac{5}{8}$
e. $\frac{9}{16} \div 3$
f. $\frac{3}{4} \div 9$
g. $14 \div \frac{7}{10}$
h. $35 \div \frac{7}{9}$
i. $17\frac{1}{2} \div 6\frac{1}{4}$
j. $54 \div \frac{6}{7}$
k. $3\frac{5}{9} \div 3\frac{1}{5}$
l. $2\frac{1}{2} \div 2\frac{1}{7}$
m. $6\frac{2}{3} \div 1\frac{3}{7}$
n. $56 \div \frac{8}{15}$
o. $15\frac{7}{8} \div 16\frac{1}{4}$

Be sure to set up the conditional statement for each problem and to follow each of the problem-solving steps.

1. The product of two factors is $15\frac{1}{8}$. One of the factors is $2\frac{3}{4}$. What is the other factor?

2. The product of two factors is $16\frac{13}{15}$. One of the factors is $7\frac{2}{3}$. What is the other factor?

3. Mike walked $12\frac{1}{4}$ miles in $3\frac{1}{2}$ hours. What was the average distance he walked per hour?

4. If $2\frac{3}{5}$ is multiplied by a certain number the product is $17\frac{1}{3}$. What is the number?

5. A pile of boards each $\frac{3}{4}$ inch thick is 8 feet high. How many boards are in the pile?

6. A boy scout troop hikes $3\frac{1}{4}$ miles per hour. How long will it take to hike 13 miles?

7. The weight of a box of books is 50 pounds. The weight in pounds of each book is $1\frac{3}{4}$. The weight of the box is 8 pounds. How many books are in the box?

8. How many shelves, each $2\frac{1}{2}$ feet long, can be cut from a board $17\frac{1}{2}$ feet long? (Disregard the width of the saw cut.)

9. The weight of a gallon of water is about $8\frac{1}{3}$ pounds. How many gallons are in a ton of water? (1 ton is 2000 pounds.)

10. The weight of a package of 500 sheets of typing paper is $1\frac{1}{4}$ pounds. How many sheets are in a box whose weight is 25 pounds?

11. Helen had $\frac{3}{4}$ of her problems correct on the test yesterday. She had 12 problems correct. How many were wrong?

12. A field is $\frac{4}{5}$ as wide as it is long. The width of the field in rods is 60. What is the length in rods?

13. Mike and Larry decided to buy a camera for \$18. Larry paid $\frac{2}{3}$ of the cost, and Mike paid the rest. What was each boy's share of the cost?

14. A television set regularly priced at \$210 was advertised at $\frac{1}{3}$ off. What was the sale price?

15. Mr. Fairfax estimated that $\frac{3}{5}$ of the total income of his hardware store was spent for buying merchandise, $\frac{1}{3}$ was spent for expenses of running the store, and the balance was profit. What fraction of the income was profit?

16. Mr. Fairfax had a total income from his store of \$6000 during August. What was his profit during that month?

A. *Multiply:*

1. $\frac{1}{8} \times \frac{3}{4}$	**4.** $\frac{5}{8} \times \frac{3}{15}$	**7.** $\frac{1}{6} \times \frac{3}{4}$
2. $\frac{3}{5} \times \frac{5}{6}$	**5.** $\frac{2}{3} \times \frac{3}{7}$	**8.** $\frac{4}{7} \times \frac{5}{8}$
3. $\frac{2}{3} \times \frac{5}{9}$	**6.** $\frac{4}{5} \times \frac{5}{6}$	**9.** $\frac{3}{5} \times \frac{10}{21}$

B. *Multiply:*

1. $32 \times \frac{1}{4}$	**4.** $\frac{5}{9} \times \$36$	**7.** $56 \times \frac{7}{8}$
2. $\frac{2}{5} \times \$25$	**5.** $42 \times \frac{3}{7}$	**8.** $\frac{2}{3} \times \$12.30$
3. $28 \times \frac{3}{4}$	**6.** $\frac{4}{5} \times \$45.50$	**9.** $42 \times \frac{5}{7}$

C. *Multiply:*

1. $15 \times 2\frac{2}{3}$	**4.** $5\frac{3}{8} \times \$16.56$	**7.** $6\frac{1}{8} \times \$24$
2. $36 \times 3\frac{5}{9}$	**5.** $4\frac{1}{2} \times \$18$	**8.** $4\frac{2}{5} \times \$30$
3. $42 \times 4\frac{5}{6}$	**6.** $8\frac{3}{4} \times \$24$	**9.** $7\frac{3}{4} \times \$16$

D. *Multiply:*

1. $5\frac{2}{5} \times 3\frac{1}{4}$	**4.** $2\frac{4}{5} \times 2\frac{1}{7}$	**7.** $3\frac{5}{8} \times 7\frac{1}{3}$
2. $4\frac{1}{5} \times 3\frac{1}{7}$	**5.** $4\frac{3}{8} \times \frac{4}{5}$	**8.** $6\frac{1}{5} \times 8\frac{1}{3}$
3. $5\frac{1}{4} \times 2\frac{4}{5}$	**6.** $14\frac{2}{7} \times 5\frac{3}{5}$	**9.** $4\frac{5}{6} \times 3\frac{1}{7}$

E. *Divide:*

1. $13 \div \frac{1}{8}$	**4.** $25 \div \frac{15}{16}$	**7.** $\frac{3}{5} \div \frac{7}{15}$
2. $6 \div \frac{1}{2}$	**5.** $\frac{9}{16} \div \frac{3}{8}$	**8.** $\frac{6}{11} \div \frac{3}{8}$
3. $16 \div \frac{16}{17}$	**6.** $\frac{3}{9} \div \frac{3}{5}$	**9.** $\frac{4}{9} \div \frac{3}{8}$

F. *Divide:*

1. $\frac{15}{16} \div 12$	**4.** $5\frac{1}{4} \div 14$	**7.** $3\frac{2}{5} \div 9$
2. $\frac{5}{8} \div 5$	**5.** $5\frac{1}{2} \div 18$	**8.** $6\frac{1}{3} \div 7$
3. $\frac{7}{8} \div 16$	**6.** $6\frac{2}{3} \div 20$	**9.** $4\frac{1}{6} \div 5$

G. *Divide:*

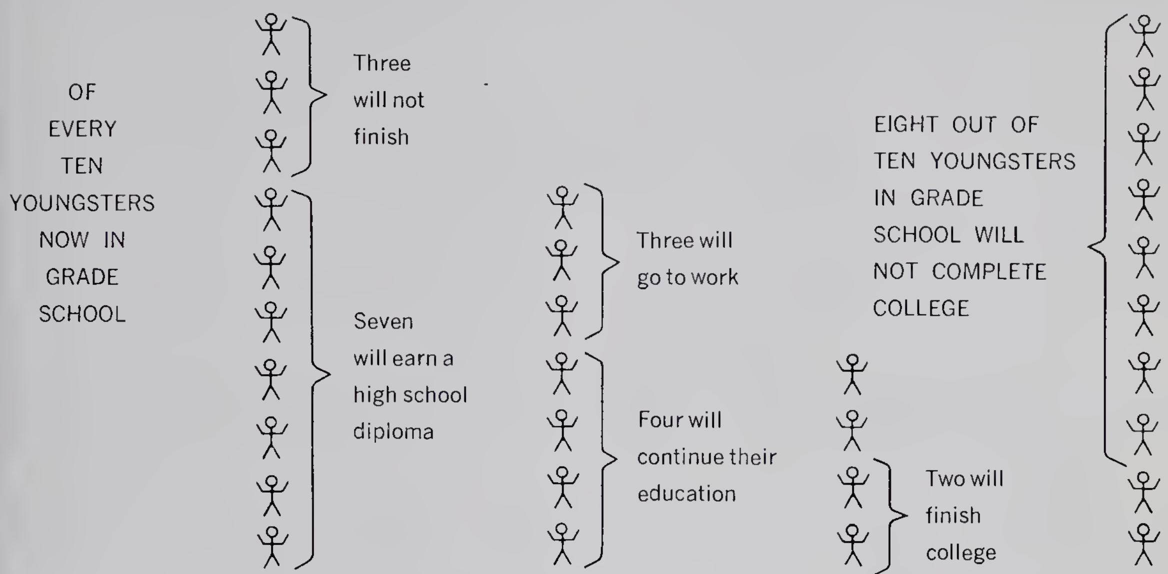
1. $9\frac{1}{3} \div \frac{7}{8}$	**4.** $7\frac{1}{2} \div 2\frac{5}{8}$	**7.** $2\frac{1}{4} \div 3\frac{3}{8}$
2. $6\frac{2}{3} \div \frac{4}{5}$	**5.** $5\frac{2}{5} \div 1\frac{1}{8}$	**8.** $5\frac{5}{16} \div 1\frac{1}{16}$
3. $6\frac{1}{4} \div \frac{15}{16}$	**6.** $7\frac{3}{5} \div 3\frac{4}{5}$	**9.** $9\frac{1}{3} \div 2\frac{2}{3}$

If you need more practice, turn to the Practice Exercises on page 448 and following. If not you may work in the Experts' Corner on the following page.

Dropouts from School

Using the graph below, answer each of the following questions.

1960-1970: THE DECADE OF EDUCATION



1. What fraction of the pupils now in grade school will not finish high school?
2. Of 800 pupils in the average grade school, how many will finish high school? How many will probably go on to college? How many will graduate from college?
3. What fraction of high school graduates will go on to college?
4. In a high school class of 420, how many will go on to college?
5. What fraction of those entering college will complete college?
6. What fraction of the pupils now in grade school are expected to complete four years of college?
7. The number of high school graduates going on to college varies among schools. In a class graduating from one high school recently, of 600 graduates divided equally between boys and girls, $\frac{7}{20}$ of the girls entered college. How many girls entered college?
8. About $\frac{9}{20}$ of the boys who graduated in the class of 600 entered college. How many might be expected to complete college?

QUESTIONS FOR RESEARCH AND DISCUSSION

1. It is a generally accepted fact that a dropout from high school "has committed economic suicide." What does this mean?
2. What are some of the advantages of a college education?

For each statement, first write the equivalent statement (if necessary) that has the variable alone on one side. Then solve and check by substituting the value for the variable in the original statement.

EXAMPLES

1. $N \times 6\frac{1}{4} = 50$ $\qquad x \times y = p$
   $N = 50 \div 6\frac{1}{4}$ $\qquad x = p \div y$
   $N = 8$
   Check: $8 \times 6\frac{1}{4} = 50$

2. $62\frac{1}{2} \div N = 25$ $\qquad p \div x = y$
   $N = 62\frac{1}{2} \div 25$ $\qquad x = p \div y$
   $N = 2\frac{1}{2}$
   Check: $62\frac{1}{2} \div 2\frac{1}{2} = 25$

1. $83\frac{1}{3} \div N = 5$
2. $N \times 16\frac{2}{3} = 150$
3. $N \times 41\frac{2}{7} = 289$
4. $72 = 3\frac{1}{2} \times N$
5. $83\frac{1}{3} \div 66\frac{2}{3} = N$
6. $18\frac{3}{4} \times N = 75$
7. $185 = 18\frac{1}{2} \times N$
8. $N \times 33\frac{1}{3} = 200$
9. $56 = N \times 8\frac{3}{4}$
10. $26\frac{1}{2} \div N = 5\frac{3}{10}$
11. $83\frac{1}{3} \times 7\frac{1}{2} = N$
12. $N = 33\frac{1}{3} \times 13\frac{3}{10}$
13. $17\frac{1}{2} = 105 \div N$
14. $N \div 83\frac{1}{3} = 3$
15. $150 = N \times 8\frac{1}{3}$
16. $6\frac{1}{4} \times 64 = N$
17. $56 \div N = \frac{1}{2}$
18. $36 = N \div 4\frac{1}{2}$
19. $48 \div N = 5\frac{1}{3}$
20. $88 = 14\frac{2}{3} \times N$
21. $12\frac{1}{2} \times N = 37\frac{1}{2}$
22. $87\frac{1}{2} \div 37\frac{1}{2} = N$
23. $13\frac{1}{7} \times 35 = N$
24. $N \div 87\frac{1}{2} = 560$
25. $49\frac{7}{8} \div N = 7\frac{1}{8}$
26. $3\frac{1}{7} \times N = 22$
27. $225 = 56\frac{1}{4} \times N$
28. $275 \div N = 15$
29. $N \div 13\frac{1}{3} = 3$
30. $37\frac{1}{2} \times N = 120$
31. $96 = 62\frac{1}{2} \times N$
32. $250 = N \div 13\frac{2}{5}$
33. $56\frac{1}{4} \div N = 42\frac{3}{16}$
34. $37\frac{1}{2} \div N = 25$
35. $66\frac{2}{3} \div 8\frac{1}{3} = N$
36. $N \times 18\frac{3}{4} = 75$
37. $N \div 13\frac{1}{3} = 27$
38. $66 = 16\frac{1}{2} \times N$
39. $6 = N \div 33\frac{1}{3}$
40. $18 \div N = 3\frac{3}{5}$
41. $81 = 4\frac{1}{2} \times N$
42. $N \times 5\frac{1}{3} = 48$
43. $50 \div N = 6\frac{1}{4}$
44. $64 = N \div 2\frac{3}{8}$

1. The sophomore class collected \$64 for the Red Cross last week. Their quota is \$120. What fraction of their quota do they have left to collect?
2. Arthur Smith, a truck driver, put in the following hours last week: Monday, $8\frac{1}{2}$; Tuesday, $9\frac{1}{4}$; Wednesday, $7\frac{3}{4}$; Thursday, $8\frac{1}{2}$; Friday, 10; Saturday, 4. What was the total number of hours he worked during the week?
3. Arthur Smith receives overtime pay for all time over 40 hours per week. How many hours overtime did he work during the week?
4. Carol worked as a baby-sitter one summer. During one week she worked $3\frac{1}{2}$ hours on Monday, $4\frac{1}{4}$ hours on Tuesday, $5\frac{3}{4}$ hours on Thursday, $4\frac{1}{3}$ hours on Saturday, and $2\frac{3}{4}$ hours on Sunday. How many hours did Carol work during the week?
5. George bought a bicycle from a friend. He agreed to pay $\frac{1}{2}$ of the total price immediately, $\frac{1}{3}$ in 30 days, and the balance in 60 days. What fraction of the total cost of the bicycle was the balance?
6. Joe, Harry, and George picked 80 bushels of apples. Joe picked $28\frac{1}{4}$ bushels, and Harry picked $35\frac{1}{2}$ bushels. How many bushels did George pick?
7. Mrs. Anderson bought a box of apricots weighing 25 pounds. If the box weighed $4\frac{1}{2}$ pounds, what was the weight of the apricots?
8. Central Bluff and Brownsville are $58\frac{1}{2}$ miles apart. If Carlsburg lies between Central Bluff and Brownsville and is $14\frac{1}{2}$ miles from Central Bluff, how far is Carlsburg from Brownsville?
9. Mr. Larsen is shipping 3 boxes. One box weighs $6\frac{1}{8}$ lb., and another $7\frac{3}{4}$ lb. The third weighs $19\frac{1}{2}$ lb. What is the total weight of the three?
10. Mrs. Lewis bought a box filled with oranges. The total weight was $46\frac{1}{2}$ pounds. If the box weighed $8\frac{2}{3}$ pounds, what was the weight of the oranges?
11. A gun weighs $7\frac{3}{4}$ pounds. When packaged for shipment the total shipping weight is $10\frac{7}{8}$ pounds. What is the weight of the packing material alone?
12. John weighed $127\frac{1}{2}$ pounds. After the first week he had gained $1\frac{1}{4}$ pounds. The second week he lost $\frac{5}{16}$ of a pound. What did he weigh after the second week?
13. Four sections of highway totaling $15\frac{2}{3}$ miles are to be built. Three of the sections measure $3\frac{1}{2}$, $5\frac{3}{5}$, and $5\frac{1}{6}$ miles. What is the length of the fourth section?

14. Four sections of highway are to be built. The four lengths measure $3\frac{1}{2}$ miles, $2\frac{1}{8}$ miles, $4\frac{3}{4}$ miles, and $5\frac{1}{6}$ miles. What is the total length of the four sections?

15. Mr. Adams had 720 bushels of wheat in storage. In January he sold $\frac{1}{3}$ of it. In March he sold $\frac{3}{4}$ of what was left. How much wheat then remained in storage?

16. Arthur Smith drove his truck from San Jose to Bakersfield, a distance of 297 miles, in $6\frac{3}{4}$ hours. What was his average hourly rate?

17. A box of books weighs 44 pounds net (without the box and packing). Each book weighs $1\frac{3}{8}$ pounds. How many books are in the box?

18. The cruising speed of a jet plane is 620 miles per hour. At that rate how far will it travel in $4\frac{3}{4}$ hours?

19. The weight of a box of apples, including the box, is 50 pounds. The box alone weighs $6\frac{1}{2}$ pounds. What is the weight of the apples?

20. A steel rod $18\frac{3}{4}$ inches long is to be cut into 3 equal pieces. What will be the length of each piece?

21. The product of two factors is $13\frac{1}{8}$. One of the factors is $1\frac{3}{4}$. What is the other factor?

22. The sum of two addends is $17\frac{1}{3}$. One of the addends is $9\frac{1}{2}$. What is the other addend?

23. A pile of planks, each $1\frac{1}{2}$ inches in thickness, is 6 feet in height. How many planks are in the pile?

24. Mr. Adams had 1800 bushels of wheat in storage. In January he sold $\frac{2}{3}$ of it. In March he sold $\frac{3}{4}$ of what was left. How many bushels remained in storage?

25. A contractor started construction of a stretch of highway $15\frac{3}{8}$ miles in length in January. By September three sections were completed that were, respectively, $3\frac{1}{2}$ miles, $5\frac{3}{4}$ miles, and $2\frac{3}{16}$ miles in length. How many miles of highway remained uncompleted?

26. At a clearance sale prices on television sets were reduced by $\frac{1}{4}$ of the regular price. One television set sold for $360. If this was $\frac{3}{4}$ of the regular price, what was the regular price of the set?

27. It is $129\frac{1}{4}$ miles from Ellsworth to Saint Cloud. The bus makes the trip in $2\frac{3}{4}$ hours. What is the average distance per hour traveled by the bus?

28. Jane types manuscripts at 60¢ a page during her spare time. Last week she kept a record of the number of hours she spent in typing, as follows: Monday, $2\frac{1}{3}$; Tuesday, $2\frac{1}{2}$; Wednesday, $1\frac{3}{4}$; Thursday, $2\frac{2}{3}$; Friday, $1\frac{3}{4}$. She earned $16.50 during the week. What were her average earnings per hour?

Estimate the answer to each of these problems before working it. Round the numbers when it will simplify your estimate. State whether your estimate is greater than the precise answer, or less. Then work the problem, and see how close you came.

1. Frying chickens are on sale at 39¢ per pound. How much will a $3\frac{1}{2}$-pound chicken cost?
2. Mr. White's car averages 16 miles per gallon of gasoline. How many gallons will he use to go 300 miles?
3. Coffee sells for 68¢ per pound. How much will be saved by buying a 3-pound can for $1.95?
4. Mr. Casper earns $400 per month. He spends an average of $\frac{1}{8}$ of his salary on transportation. How much does he spend on transportation?
5. The boys who play backfield for Westlake High School weigh 159, 183, 165, and 177 pounds respectively. What is their total weight?
6. A new jet passenger plane averaged 590 miles per hour for $3\frac{1}{4}$ hours. How far did it travel?
7. A load of 31 pigs weighed 6107 pounds. What was the average weight of the pigs?
8. Mr. Pyle paid $58.90 for a set of four tires. What was the price per tire?
9. Bob wants to buy a bicycle priced at $69.95. He has saved $31.25. How much more money does he need to buy the bicycle?
10. Find the cost of a half dozen basketballs at $8.90 each.
11. Merrill Junior High School has 214 pupils in the seventh grade, 206 pupils in the eighth grade, and 217 in the ninth grade. What is the total enrollment at the school?
12. Jim bought 5 gallons of gasoline at 39¢ a gallon. How much change should he receive from a $5 bill?
13. The Boy Scouts hiked 16 miles in $5\frac{1}{2}$ hours. What was their average rate in miles per hour?
14. Mike is going to cut a steel bar $26\frac{1}{4}''$ long into pieces each $3\frac{1}{2}''$ long. How many pieces will he get?
15. The Andersons are planning to leave at 8:30 A.M. to drive to Elmtown which is 237 miles away. They will average 40 miles per hour. At what time should they arrive?
16. If $\frac{1}{10}$ of a yearly income is withheld for income tax, how much of a $6,000 income is withheld?

You will remember from your earlier study that the Hindu-Arabic system of numeration is base ten. Consider the numeral 1111 and let us review what it means.

1. What does the digit in the place farthest to the right stand for?
2. What is the value of the digit in the place farthest to the left?
3. Which place is the *tens* place?
4. As we move from the far left to the right, is the value of each place in 1111 equal to $\frac{1}{10}$ of the value of the place preceding it?

$$\begin{array}{r} 1 \times 1000 = 1000 \\ 1 \times 100 = 100 \\ 1 \times 10 = 10 \\ 1 \times 1 = \underline{1} \\ 1111 \end{array}$$

5. A mathematician who lived about the time the Pilgrims settled in North America devised a method whereby our number system could be extended to the right of the ones place. To be consistent, how would the next place to the right compare to the ones place in value?
6. To extend the system to the right of the ones place the mathematician needed some symbol to identify the ones place. Early writers used a dot below the figure in the ones place for this purpose. Using that system, what figure is in the ones place in the numeral $7\underset{\cdot}{6}5$?
7. Later on, in error, a printer placed the point, which we will call *decimal point*, to the right of the figure in the ones place where it has been ever since. Write the numeral in Exercise 6 as we would write it today.

Fractions whose denominators are powers of 10, for example, 10 (10^1), 100 (10^2), 1000 (10^3), etc., can be written with a decimal point as follows:

$$\frac{1}{10} = .1 \qquad \frac{8}{10{,}000} = .0008$$

$$\frac{3}{100} = .03 \qquad \frac{4}{100{,}000} = .00004$$

$$\frac{6}{1000} = .006 \qquad \frac{6}{1{,}000{,}000} = .000006$$

Fractions written with a decimal point are called *decimals*. Since the set of whole numbers is a subset of the set of fractional numbers, then whole numbers can also be written with a decimal point. However,

we omit the decimal point when we write a numeral that names a whole number, such as 0, 1, 2, 3, etc.

In writing decimals that name fractional numbers less than 1, it may be helpful to write a zero in the ones place. For example, .19 might better be written as 0.19. This avoids possibilities of error by calling attention to the decimal point.

Notice that one digit to the right of the decimal point, such as .1, .2, .3, etc., represents tenths, two digits, such as .11, .29, .06, represent hundredths, three digits, such as .018, .003, .502, etc., represent thousandths and four digits, such as .0008, .0104, .2007, etc., represent ten-thousandths. Notice that in decimals such as .0008, the zeros are place holders.

Examine the table below as it will help you to read and write decimals. The table shows the names of the places to the right of the decimal point up to millionths. Of course, the table could go on to ten-millionths, hundred-millionths, etc.

millions	hundred thousands	ten thousands	thousands	hundreds	tens	ones		tenths	hundredths	thousandths	ten-thousandths	hundred-thousandths	millionths
					3	2	.	3	6	9	0	6	5

Can you read the numeral in the table?

The names for the positions to the right of the decimal point are easy to remember because they relate to fractions. Notice how each of the following is read.

a. 0.005 is read five thousandths

$\frac{5}{1000}$ is read five thousandths

b. 0.0009 is read nine ten-thousandths

$\frac{9}{10,000}$ is read nine ten-thousandths

c. 2.004916 is read two *and* four thousand nine hundred sixteen millionths just as $2\ \frac{4916}{1,000,000}$ is read

In reading a decimal that has digits to the *left* of the decimal point, read the decimal point as "and."

8. Write each of the following as a decimal and as a fraction.

a. One thousand four hundred and six hundredths
b. Four hundred nine thousandths
c. Sixty and four tenths
d. Twenty-eight thousand thirty-six
e. Four million, seven hundred sixty-five thousand, seven hundred nine and five tenths
f. Twelve hundredths
g. Two billion, seven hundred six thousand, two hundred seventy-eight
h. Eighteen ten-thousandths
i. Six and six thousandths
j. Thirty-two and one hundred one ten-thousandths

9. Write each of the following as a decimal.

a. $\frac{3}{10}$	**e.** $\frac{77}{10,000}$	**i.** $5\frac{7}{100}$
b. $\frac{6}{1000}$	**f.** $\frac{89}{100}$	**j.** $2\frac{32}{1000}$
c. $\frac{45}{1000}$	**g.** $\frac{4}{10}$	**k.** $56\frac{3}{10,000}$
d. $\frac{1}{1000}$	**h.** $\frac{23}{10,000}$	**l.** $141\frac{402}{10,000}$

10. Write each of the following as a fraction in simplest form.

a. 0.13	**e.** 0.95	**i.** 2.573
b. 0.5	**f.** 0.005	**j.** 40.04
c. 0.045	**g.** 0.0032	**k.** 10.0001
d. 0.64	**h.** 0.1095	**l.** 729.729

11. In studying whole numbers in Chapter 1 you learned that the value of each digit in our number system depends on the place it holds. This is true of decimals as well as whole numbers. As we move to the right of the decimal point, each place must have a value equal to $\frac{1}{10}$ of the value to its left. You can readily see this if you set up a table to show the value of each digit in a numeral.

Let's examine 7.7777.

$$\begin{aligned}
7 &= 7 \times 1 \\
.7 &= 7 \times (\tfrac{1}{10} \times 1) = 7 \times \tfrac{1}{10} \\
.07 &= 7 \times (\tfrac{1}{10} \times \tfrac{1}{10}) = 7 \times \tfrac{1}{100} \\
.007 &= 7 \times (\tfrac{1}{10} \times \tfrac{1}{100}) = 7 \times \tfrac{1}{1000} \\
.0007 &= 7 \times (\tfrac{1}{10} \times \tfrac{1}{1000}) = 7 \times \tfrac{1}{10,000}
\end{aligned}$$

Make a similar chart to examine each of the following:

a. 77.777	**d.** 123.006	**g.** 62.384
b. 1038.002	**e.** 43.148	**h.** 4.00798
c. 5.021	**f.** 7.3984	**i.** 12.60049

In adding or subtracting whole numbers you write the numerals in columns, being careful to write all the ones digits in the same column, as well as the digits in the tens, hundreds, etc., places. Similarly, in adding or subtracting using decimals, we must be careful to write the tenths, hundredths, etc., digits in their respective columns. This is readily done if you align the decimal points.

Before you add or subtract, you must be careful to see that each numeral is expressed to the same number of places after the decimal point.

EXAMPLES

1. If the numerals represent money or counted objects, annex as many zeros as necessary to bring each decimal to the same degree of precision.

a. 0.65		0.650		**b.** $3.15		$3.15
3.2	becomes	3.200		5	becomes	5.00
0.513		0.513		4		4.00
		4.363				$12.15

2. If the numerals represent measurements in inches, pounds, etc., round each number down to the least degree of precision. The digit in the hundredths position determines whether you round up or down in the tenths position. Similarly, the digit in the tenths position determines whether you round up or down in the ones position. This follows the same basic procedure as rounding whole numbers (as discussed in Chapter 2). In each of the following, the least degree of precision is determined by the numeral having the least number of digits to the right of the decimal point.

a. 10.833		10.8		**b.** 16.36		16
6.25	becomes	6.3		44.522	becomes	45
0.8		0.8		7		7
		17.9				68

1. Each of the following represents measurement. Round and add.

a.	**b.**	**c.**	**d.**
14.023	0.35	143.64	3.261
3.98	49.995	0.796	61.7608
4.1	8.001	55.6648	9.0182

2. Each of the following represents measurement. Round and subtract.

a.	**b.**	**c.**	**d.**
3.175	9.4	111	6.5
− 1.5	− 3.351	− 83.5	− 0.625

Casting out nines provides a useful check on the accuracy of your computations with decimals as well as with whole numbers. As you can see in the Example, the decimal point is ignored.

In the Example at the right, the sum of the digits is found for each addend, and the excess over a multiple of 9 is listed under Excesses. Inspection will show that the sum of the excesses, after removing the largest multiple of 9, is 4. This is equal to the excess from the sum of the digits in the answer, 266.26. Since they agree, the answer is *probably* right. What kind of mistake will this check *not* reveal?

EXAMPLE

Add:	*Excesses*
8.05	4
19.62	0
38.47	4
.05	5
200.07	0
266.26	4

In checking subtraction, the excess in the minuend less the excess in the subtrahend, should equal the excess in the difference. If the excess in the minuend is less than that in the subtrahend, 9 is added to the excess of the minuend. In the Example you see that the excess from the difference equals the difference of the excesses from the minuend and subtrahend.

EXAMPLE

Subtract:	*Excesses*	
18.012	3 + 9 =	12
− 11.625	6	− 6
6.387	6	6

Check the following computations by casting out 9's. Round the numbers as necessary.

Add:

1. 38.67	**3.** 39.21	**5.** 0.07 in.	**7.** 5280 lb.
15.43	7.3	4.051 in.	341.9 lb.
7.19	117.095	11.625 in.	27 lb.
95.68	8.9	9.5 in.	6.8 lb.
16.07	55.95	6.25 in.	39.1 lb.

2. $ 19.53	**4.** 517.831	**6.** $ .65	**8.** 6.21 gal.
6	65.545	3.85	4.032 gal.
108.56	100.083	15	9 gal.
13.65	18.125	32.04	17.6 gal.
10	87.225	6	7.92 gal.

Subtract:

9. 45.625	**10.** $90	**11.** 107.5 lb.	**12.** 67.895 mi.
− 27.009	− 67.83	− 86.75 lb.	− 65.5 mi.

A. *Addition:* Copy and add; round when necessary.

1. 4.25 + 125.7 + 13.008 + .9
2. 16 + .07 + 28.7 + 4.83
3. 57.07 in. + 5.9 in. + 28.602 in. + .0775 in.
4. 35.9 lb. + 28 lb. + .052 lb. + 69.06 lb. + 38.859 lb.

B. Express in decimal form, and add.

1. 506 ten-thousandths + 4 thousandths + 9 tenths
2. 32 hundredths + 63 ten-thousandths + 45 thousandths
3. 1431 ten-thousandths + 19 thousandths + 30 thousandths
4. 7 hundredths + 306 thousandths + 18 ten-thousandths
5. 156 thousandths, 147 ten-thousandths, and 14 hundredths
6. 95 thousandths, 427 ten-thousandths, and 53 hundredths
7. 155 thousandths, 7 ten-thousandths, and 57 ten-thousandths
8. 15 thousandths, 233 hundred-thousandths, and 13 hundredths
9. 3232 ten-thousandths, 75 thousandths, and 505 ten-thousandths

C. *Subtraction:* Copy and subtract; round when necessary.

1. 7.348 – 5.135
2. 25.562 – 17.281
3. 16.303 – 9.58
4. 18.5 lb. – 14.25 lb.
5. 38.03 ft. – 19.555 ft.
6. 16.25 in. – 9.507 in.

D. Express in decimal form, and subtract.

1. 19 hundredths from 321 thousandths
2. 135 thousandths from 5 tenths
3. 3 hundredths from 75 thousandths
4. 75 thousandths from 932 ten-thousandths
5. 25 ten-thousandths from 7 tenths
6. 1839 ten-thousandths from 995 thousandths
7. 25 hundredths from 715 thousandths
8. 7761 ten-thousandths from 853 thousandths
9. 8 hundredths from 11 hundredths

If you need more practice, turn to the Practice Exercises on page 452 and following. If not, you may work in the Experts' Corner on the following page.

Using Large Numbers: Scientific Notation

Reading and writing names for large numbers is sometimes very difficult. The following problems illustrate a simple way of naming these large numbers.

1. *Scientific notation* is naming a number as a product of two factors. One factor is greater than or equal to 1 but less than 10. The other factor is a power of 10. The numeral 25,000,000,000,000 can be written in scientific notation as 2.5×10^{13}. Here the exponent 13 tells how many times 10 is used as a factor. Use scientific notation to write the number named in this statement: Recently the national debt was announced as being $275,000,000,000.

2. Each of the numbers named in the following table can be written simply as a power of 10. This is called the *exponential form* of the number. For example, 10^6 is the exponential form for one million. Copy and complete this table.

In words	*Standard form*	*Exponential form*
a. One million	1,000,000	10^6
b. One billion	1,000,000,000	?
c. One trillion	?	?
d. One decillion	?	?

3. A scientist has estimated that the total number of atoms in the universe is $3 \times (10)^{74}$. If you were to write this in standard form, how many zeros would the numeral have?

4. It has been calculated that there are 40,000,000,000,000,000 tons of salt in the ocean. Write this numeral in words and also in scientific notation.

5. Calculate the weight of the salt in pounds, and write the answer in words and also in scientific notation.

6. The speed of light is about 186,000 miles per second. How far does light travel in one minute? How far does light travel in one hour? Write your answers also in scientific notation.

7. To the nearest billion miles, how far does light travel in a 24-hour day? Write your answer in scientific notation.

8. Distances between the stars are so great that instead of using miles, astronomers use the *light-year* as a unit of distance. This is the distance light will travel in a year. How far is this, to the nearest trillion miles?

Multiplication using decimals is done in the same way as with whole numbers. There is, however, the additional problem of locating the decimal point.

EXAMPLE

To locate the decimal point, we decide about how great the product should be; 3.3 is about 3 and 4.5 is about 5. Therefore, the product is about 15. Looking at 1485, we see that the decimal point should be placed after the 4.

$$\begin{array}{r} 3.3 \\ \times\, 4.5 \\ \hline 165 \\ 132 \\ \hline 14_{\wedge}85 \end{array}$$

Notice that in the above Example, each factor, 3.3 and 4.5, has one digit to the right of the decimal point. The product, after we have located the decimal point, has two digits to the right of the decimal point.

EXAMPLE

In this Example, we note that 5.7 is about 6, and 0.63 is about $\frac{1}{2}$. Therefore, the product is about 3. The decimal point should be placed after the 3 in the product.

$$\begin{array}{r} 5.7 \\ \times\, 0.63 \\ \hline 171 \\ 342 \\ \hline 3_{\wedge}591 \end{array}$$

In this Example, the factor 5.7 has one digit to the right of the decimal point, and 0.63 has two digits to the right of the decimal point. The product has 3 digits to the right of the decimal point. Does this suggest a rule for locating the decimal point in the product?

1. Use rounded numbers to help locate the decimal point in each of the following:

a. $8.35 \times 0.9 = 7515$ **c.** $2.75 \times 0.15 = 4125$ **e.** $6.53 \times 0.127 = 82931$
b. $0.75 \times 6.3 = 4725$ **d.** $0.413 \times 17.5 = 72275$ **f.** $9.9 \times 1.35 = 13365$

Did you discover the relationship between the number of digits to the right of the decimal point in the product and the number of digits to the right of the decimal point in each of the two factors?

Rule: The number of digits to the right of the decimal point in the product is equal to the sum of the number of digits to the right of the decimal point in each of the two factors.

2. Use the Rule at the bottom of page 149 to locate the decimal point in each of the following:

a. $2.5 \times 13.5 = 3375$
b. $0.025 \times 0.135 = 3375$
c. $0.25 \times 1.35 = 3375$
d. $2.5 \times 135 = 3375$
e. $25 \times 13.5 = 3375$
f. $0.25 \times .0135 = 3375$

3. Find the products.

a. $.35 \times .06$
b. 16.15×2.05
c. 62.5×1.78
d. 28.7×3.19
e. $6.23 \times .08$
f. 26.9×6.07
g. 20.08×0.67
h. 2.09×0.078
i. 5.25×0.37
j. 0.85×3.092
k. 140×0.256
l. 2.053×0.016

Be sure to set up the conditional statement before attempting to solve each of the following problems.

4. The Monroe High School basketball team has played 24 games and has won .875 of them. How many has it won?

5. A truck driver traveled 51.8 miles per hour for 8.5 hours. How far did he travel during that time?

6. During a sale at the Sport Shop all prices were reduced by 0.2 of the prices at which the goods were marked. What will a golf club sell for that is marked $15?

7. During a season the Chicago White Sox won .625 of the games they played. They played 160 games. How many did they win?

8. Jim purchased 13.5 gallons of gasoline at 33.9¢ per gallon. What did he pay for the gasoline? Figure the cost to the nearest cent.

9. At a sale a tire regularly priced at $25.00 was sold for 0.7 of its regular price. What did it sell for?

10. The Rogers family income is $6500. Of this, 0.3 is allowed for food, and 0.25 for rent. How much is allowed for each?

11. In 1940 the population of Centerville was 12,764. In 1960 the population was 1.75 as great. What was the population in 1960?

12. On a test of 30 problems Jane had 0.7 of them correct. How many did she have wrong?

13. A jet plane traveled at a speed of 560 miles an hour for 4.75 hours. How far did it travel?

14. The supermarket purchased 1200 bushels of potatoes for $2400. They were sold for 1.6 times the cost. How much did they sell for per bushel?

15. A carpenter earns $4.80 per hour. His helper earns 0.625 as much. How much per hour does the helper earn?

A fraction expresses indicated division. That is, $\frac{5}{8}$ is another way of writing $5 \div 8$.

In the Example at the right you are asked to divide 42 by 1.25. This can be expressed as $\frac{42}{1.25}$. We can find an equivalent fraction with a natural number as denominator. To do this, we multiply $\frac{42}{1.25} \times \frac{100}{100}$, since $\frac{100}{100} = 1$, which is the multiplicative identity. The product, $\frac{4200}{125}$, now makes it possible to perform the division with whole numbers. This also illustrates that fractions whose denominators are *not* powers of 10 can also be written as decimals.

EXAMPLE

Divide:

$42 \div 1.25 = ?$

$$\frac{42}{1.25} \times \frac{100}{100} = \frac{4200}{125}$$

$$\frac{4200}{125} = 125\overline{)4200}$$

1. State what you would multiply by to change each of the following to an equivalent form with a whole number as the divisor.

a. $86 \div 1.9$
b. $65 \div 2.05$
c. $93.5 \div 0.205$
d. $17.51 \div 4.9$
e. $20.75 \div 1.705$
f. $3.62 \div 0.042$

In the Example at the right, division is completed. To keep in mind what the dividend and divisor were at the start, we use a mark called the *caret* (‸) to show how many places the decimal point has been "moved" to make the divisor a whole number.

How many places was the decimal point moved in the Example?

This means that both dividend and divisor were multiplied by what number?

Notice that the decimal point in the quotient was placed over the caret in the dividend. Why is this correct?

EXAMPLE

Divide:

$42 \div 1.25 = ?$

```
           33.6
 1.25‸)42.00‸0
        37 5
         4 50
         3 75
           750
           750
```

2. Find the quotient of each of the following:

a. $12.8 \div 0.32$
b. $1.56 \div 1.2$
c. $3.43 \div 4.9$
d. $1.92 \div 3.2$
e. $17.28 \div 0.24$
f. $15.75 \div 4.5$
g. $36 \div 4.8$
h. $2.4 \div 0.64$
i. $4.9 \div 5.6$
j. $51 \div 0.136$
k. $30 \div 9.6$
l. $20.7 \div 1.84$
m. $0.87 \div 2.9$
n. $8.06 \div .62$
o. $0.64 \div 0.4$
p. $6.6 \div 1.32$
q. $17.86 \div 4.7$
r. $1.27 \div 0.254$

To avoid mistakes in locating the decimal point in a division computation, it is a useful practice to make an estimate, deciding about what the quotient should be. Ask these questions, referring to the Example at the right.

a. Is the quotient more than 1? Explain.
b. Is it more than 10? Yes. Explain.
c. Is it more than 100? Yes. Explain.
d. Is it more than 1000? No. Explain.

EXAMPLE

$$\begin{array}{r} ? \\ 3.55\overline{)621.25} \end{array}$$

Estimate:

$1000 > q > 100$
q is about 150

Then the quotient, q, is between 100 and 1000, or

$$1000 > q > 100.$$

Using 4 as a trial divisor, the quotient is about 150. Complete the division, and see how close the estimate is.

In the Example below, you are to carry the division to a specified number of places. The correct procedure is to carry the division one place beyond what is required, and then round to the required position.

EXAMPLE

Find to the nearest thousandth: $12.7 \div 3.6 = ?$

$$\begin{array}{r} ? \\ 3.6_{\wedge}\overline{)12.7_{\wedge}} \end{array}$$

Estimate: $10 > q > 1$

Using 4 as a trial divisor, q is about 3.

$$\begin{array}{r} 3.5277 \\ 3.6_{\wedge}\overline{)12.7_{\wedge}0000} \\ \underline{10\ 8} \\ 1\ 9\ 0 \\ \underline{1\ 8\ 0} \\ 1\ 00 \\ \underline{72} \\ 280 \\ \underline{252} \\ 280 \\ \underline{252} \end{array}$$

What is the final answer to this Example? In the Example both dividend and divisor were multiplied by what number? What estimate was made to guide in locating the decimal point in the quotient?

Note that you can annex as many zeros as necessary after the decimal point; 127 and 127.0000 name the same number.

3. Find the quotients to the nearest hundredth:

a. $4.280 \div 0.2$	**c.** $36.03 \div 0.8$	**e.** $0.468 \div 0.13$
b. $29.6 \div 4.1$	**d.** $62.4 \div 0.8$	**f.** $24.42 \div 2.1$

4. Find the quotients to the nearest thousandth:

a. $10.35 \div 5.1$ c. $6.095 \div 5$ e. $4.872 \div 2.4$
b. $1.936 \div 1.6$ d. $6.7 \div 2.8$ f. $29.8 \div 18$

Examine the Rule below and see how it is applied in the Example which follows it.

Rule: To find the decimal equivalent of a fraction, divide the numerator by the denominator.

EXAMPLE

Find the decimal equivalent of $\frac{5}{8}$.

By definition, $\frac{5}{8} = 5 \div 8$.

Since $5 < 8$, $q < 1$. Also, $q > 0.1$.

$$\begin{array}{r} .625 \\ 8\overline{)5.000} \\ \underline{4\ 8} \\ 20 \\ \underline{16} \\ 40 \\ \underline{40} \end{array}$$

---

5. Find the decimal equivalent for each of the following. If there is still a remainder after carrying the division four places, round the quotient to the nearest thousandth.

a. $\frac{3}{8}$ c. $\frac{14}{25}$ e. $\frac{8}{15}$ g. $\frac{28}{25}$ i. $\frac{5}{4}$
b. $\frac{5}{16}$ d. $\frac{2}{5}$ f. $\frac{7}{9}$ h. $\frac{7}{4}$ j. $\frac{19}{20}$

EXAMPLE

Find the quotient $\$95.18 \div 17$, to the nearest cent.

$$\begin{array}{r} 5.59\frac{15}{17} = \$5.60 \\ 17\overline{)\$95.18} \\ \underline{85} \\ 10\ 1 \\ \underline{8\ 5} \\ 1\ 68 \\ \underline{1\ 53} \\ 15 \end{array}$$

How much was the remainder?
Was it more or less than half of the divisor?
Why did we call the answer \$5.60 instead of \$5.59?
What would the answer have been if the remainder had been 7 instead of 15?
This is an alternate method for rounding a quotient.

---

6. Find the quotients to the nearest cent by the above method.

a. $\$80.28 \div 17$ c. $\$90.50 \div 16$ e. $\$90.68 \div 55$
b. $\$74.20 \div 35$ d. $\$35.56 \div 14$ f. $\$20.60 \div 15$

Numerals in Exponential Form

Earlier in this chapter you expressed the meaning of digits in a numeral by using powers of 10. When a power of 10 is indicated by an exponent the numeral is said to be in *exponential form.* Thus, expressed in exponential form: $100 = 10^2$; $1000 = 10^3$; $3000 = 3 \times 10^3$; etc. As we shall see, it is possible to use exponential form in expressing the meaning of digits to the right of the decimal point, as well as to the left of it.

1. Write the following numerals in exponential form.
a. 1,000,000 **b.** 10,000 **c.** 1,000,000,000

2. Write the following without using exponents.
a. 10^5 **b.** 10^8 **c.** 10^1

3. As the exponent of 10 increases by 1, the previous number is multiplied by 10. As the exponent of 10 decreases by 1, the previous number is divided by 10. Write these quotients and products without exponents.
a. $10{,}000 \times 10$ **b.** $10{,}000 \div 10$ **c.** $10^4 \times 10$ **d.** $10^4 \div 10$

4. Write the numeral expressing the value of the quotient $10^1 \div 10$ in exponential form.

5. Without the use of an exponent write the answer: $10^1 \div 10 = ?$

6. What is the value of each of the following?
a. 3×10^0 **b.** 7×10^0 **c.** 18×10^0

7. Can the exponent of 10 be a numeral less than zero? If a valley is one foot below sea level, its altitude can be expressed as $^{-}1$ foot. If the thermometer stands at 0°, and the temperature falls 1°, the thermometer reads $^{-}1°$. To carry the rule expressed in Exercise 3 one step further, if we divide 10^0 by 10 the quotient should be 10^{-1}. That is, $10^0 \div 10 = 10^{-1}$. Express this division as a fraction: $1 \div 10 = ?$

8. Write the answer as a decimal: $1 \div 10 = ?$

9. Write the following numerals as decimals.
a. 4×10^{-1} **b.** 5×10^{-1} **c.** 11×10^{-1} **d.** 15×10^{-1}

10. Write the following numerals in exponential form.
a. 0.4 **b.** 0.6 **c.** 0.5 **d.** 1.8 **e.** 17.9 **f.** 15.2

11. To determine the meaning of 10^{-2} we can divide: $10^{-1} \div 10 = 10^{-2}$ Write the value of 10^{-2} as a fraction and as a decimal.

12. Write the following as decimals.
a. 18×10^{-2} **b.** 27×10^{-2} **c.** 11×10^{-2} **d.** 123×10^{-2}

13. Extending the rule of Exercise 3 you can determine the meaning of 10^{-3} by dividing 10^{-2} by 10. Write out the division, using exponential form, and also without exponents.
14. Write as numerals with negative exponents.
    a. 0.2 **b.** 0.5 **c.** 0.17 **d.** 0.117 **e.** 0.23
15. Write as decimals: 10^{-4}; 10^{-5}; 10^{-6}.
16. Write in exponential form.
    a. 0.175 **b.** 0.1215 **c.** 0.0125 **d.** 0.0037
17. Using negative exponents it is possible to analyze decimals to indicate the value of the number named by each digit. Thus, 325.1635 can be analyzed as follows:

$$\begin{aligned}
3 \times 10^2 &= 300 \\
2 \times 10^1 &= 20 \\
5 \times 10^0 &= 5 \\
1 \times 10^{-1} &= 0.1 \\
6 \times 10^{-2} &= 0.06 \\
3 \times 10^{-3} &= 0.003 \\
5 \times 10^{-4} &= \underline{0.0005} \\
& 325.1635
\end{aligned}$$

Using this procedure, analyze 705.04013.

18. Scientific notation is useful in expressing large numbers. The astronomer thinks of the distance to Proxima Centauri, the nearest star to our sun, as 2.5×10^{13} miles. Write this numeral without using exponents.
19. The sun is approximately 93,000,000 miles from the earth. Expressed in scientific notation this is $9.3 \times 10^{?}$.
20. The distances in space are so great that they are difficult to comprehend. The distance to Proxima Centauri can be understood if you know that light from the sun reaches us in approximately 8 minutes. Light from Proxima Centauri takes 4.2 years to reach the earth. Express 4.2 years in minutes. (Use 365 days = 1 year.)
21. Round your answer to Exercise 20 to the nearest million and write the numeral in scientific notation.
22. We can use scientific notation also to simplify the expression of very small numbers. The diameter of a uranium atom is 3.5×10^{-8} cm. Write this numeral as a fraction.
23. The smallness of an atom is also hard to comprehend. What is the diameter of a molecule that is a million times larger than a uranium atom? ($3.5 \times 10^{-8} \times 1{,}000{,}000 = ?$)
24. The diameter of a hydrogen atom is 0.53×10^{-8} cm. Write this numeral as a fraction.

You have found that in checking addition and subtraction, you can ignore the decimal point in casting out nines. This is also true with multiplication and division. In the following Example, note that the product of the excesses in the factors equals the excess in the product. This indicates that the answer is *probably* correct. How can you check on the correctness of the location of the decimal point?

EXAMPLE

$$\begin{array}{r} 4.82 \\ 1.21 \\ \hline 4\ 82 \\ 96\ 4 \\ 482 \\ \hline 5.8322 \end{array}$$

$4.82 = 5;$ (Excess); $1.21 = 4;$ (Excess);

$$\begin{array}{r} 5 \\ \times\ 4 \\ \hline 20 \end{array} = 2 \text{ (Excess)}$$

$5.8322 = 20;\ \ 20 = 2$ (Excess)

In using casting out nines for checking computations in division, we use the relationship: divisor $\times$ quotient + remainder = dividend. Instead of making the check with the original numbers, we can use the excesses as in the following Example.

EXAMPLE

$$\begin{array}{r} 5.96 \\ 6.4_{\wedge}\overline{)38.1_{\wedge}99} \\ 32\ 0 \\ \hline 6\ 1\ 9 \\ 5\ 7\ 6 \\ \hline 4\ 39 \\ 3\ 84 \\ \hline 55 \end{array} = \text{Remainder}$$

$6.4 = 1$ (Excess); $5.96 = 2$ (Excess);

$$\begin{array}{r} 1 \\ \times\ 2 \\ \hline 2 \\ +\ 1 \\ \hline 3 \end{array} \text{ (Excess)}$$

$55 = 1$ (Excess);

$38.199 = 3$ (Excess)

Why must you check before rounding the quotient? Will casting out nines provide the correct check for locating the decimal point?

Perform the indicated operations and check by casting out nines. Carry division to the nearest hundredth, or to the nearest cent.

1. 15.7×8.33
2. 0.13×9.55
3. 25.34×92.76
4. 6.075×9.7
5. 17.5×9.24
6. 9.75×0.0875
7. $3.47 \div 0.15$
8. $44.55 \div 25.9$
9. $17.85 \div 31.3$
10. $56.43 \div 87.71$
11. $\$47.50 \div 13$
12. $\$735.45 \div 25.7$

Write the conditional statement for each problem before undertaking the solution. First estimate the answer for each computation.

1. A steel rod 33 inches long is to be cut into pieces each 2.75 inches long. If you disregard the width of the saw cut, how many pieces will there be?
2. The basketball team has played 16 games and has won 10 of them. Express as a fraction and as a decimal what part of its games the team has won.
3. A plane made a flight of 3437.5 miles in 6.25 hours. What was the average number of miles per hour it traveled?
4. A boy scout troop on a hike averages 3.5 miles per hour. How long will it take them to travel a distance of 12.25 miles at that rate?
5. There are 920 students in the Jefferson High School. Of these 375 are in the ninth grade. Express as a fraction in simplest form and as a decimal rounded to the nearest hundredth, what part of the students are in the ninth grade.
6. A pile of sheet metal strips each 0.375 inches thick is 18 inches high. How many strips are in the pile?
7. In many wholesale book houses an order of books, instead of being counted, will be weighed. If a book weighs 1.25 pounds, how many books are in a pile that weighs 27.50 pounds?
8. Last Saturday Mike was paid $4.95 for working 4.5 hours. How much per hour was he paid?
9. At a "month-end" sale a camera, regularly priced at $45, was sold for $38.25. The reduction was what part of the regular price? (Express the result both as a fraction and as a decimal.)
10. Henry bought a motorcycle priced at $360. He paid 0.35 of the price in cash, and the rest in 9 equal monthly payments. How much was each payment?
11. Mr. Henderson pays $1800 rent each year on the family home. He says this is 0.3 of his income. How much is his income?
    Note: The conditional statement is: $0.3 \times n = 1800$ Why?
12. When the Erickson family started on a 630-mile trip the gasoline tank was full and the odometer reading was 26,395.4. When they stopped for gasoline the reading was 26,504.9. It took 7.3 gallons to refill the tank. If the average number of miles traveled per gallon remains the same, what will the gasoline for the 630-mile trip cost at 32.9¢ per gallon?

Multiplication and Division: Decimals

A. Copy each product, and locate the decimal point.

1. $0.16 \times 0.3 = 48$
2. $3.2 \times 13 = 416$
3. $2.09 \times 1.2 = 2508$
4. $0.5 \times 0.7 = 35$
5. $2.5 \times 0.25 = 625$
6. $0.39 \times .08 = 312$
7. $9 \times 0.17 = 153$
8. $0.5 \times 0.15 = 75$
9. $1.3 \times .07 = 91$
10. $3.19 \times 2.6 = 8294$
11. $12.85 \times .015 = 19275$
12. $70.5 \times 2.94 = 207270$

B. Find the products.

1. 1.3×0.4
2. 3.9×0.3
3. $3.1 \times .07$
4. 7.0×0.16
5. $.056 \times 8$
6. 0.52×0.9
7. 6.8×8.6
8. 0.687×4.5
9. 5.3×0.76
10. 5.11×6.08
11. 32.9×18.2
12. $.05 \times .009$

C. Copy each quotient, and locate the decimal point.

1. $24.6 \div 6 = 41$
2. $8.48 \div 40 = 212$
3. $5.0 \div 25 = 2$
4. $3.64 \div 1.4 = 26$
5. $3.366 \div 0.6 = 561$
6. $6.00 \div 0.75 = 8$
7. $0.56 \div 1.4 = 4$
8. $22.5 \div .09 = 25$
9. $7.29 \div 2.7 = 27$
10. $7.2 \div .09 = 8$
11. $2.16 \div 6 = 36$
12. $81 \div .003 = 27$

D. Find the quotients.

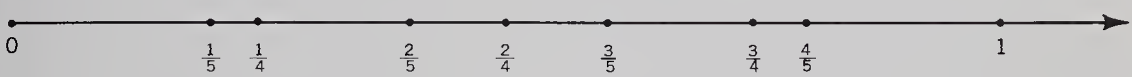
1. $1.5 \div 3$
2. $4.2 \div 6$
3. $9.1 \div .07$
4. $0.35 \div 0.7$
5. $8.4 \div .012$
6. $1.8 \div 0.6$
7. $4.242 \div 7$
8. $23.94 \div 6$
9. $56.48 \div 0.16$
10. $.0378 \div 1.4$
11. $256 \div 3.2$
12. $54 \div .025$

If you need more practice, turn to the Practice Exercises on page 456 and following. If not, you may work in the Experts' Corner on the following page.

Density of Fractions

The set of fractional numbers includes the natural numbers as a subset, and it also includes zero, which can be named as a fraction with zero as the numerator ($\frac{0}{1}$, $\frac{0}{2}$, $\frac{0}{3}$, etc.). Between any two whole numbers we can find a fractional number. Can we always find a fractional number between any two given fractional numbers?

1. Represent $\frac{3}{4}$ and $\frac{4}{5}$ on a number line. Express each fraction with denominator 40. Name a fractional number midway between $\frac{3}{4}$ and $\frac{4}{5}$.



2. Locate $\frac{3}{4}$, $\frac{4}{5}$, and $\frac{31}{40}$ on a number line. Express all three fractions with denominators 80. What fractional number is midway between $\frac{3}{4}$ and $\frac{31}{40}$? Between $\frac{31}{40}$ and $\frac{4}{5}$?

3. A general method for finding a fractional number between a given pair of fractional numbers is to find their average — that is, one-half of their sum. What number is half-way between $\frac{7}{9}$ and $\frac{8}{9}$?

$$\frac{7}{9} + \frac{8}{9} = \frac{15}{9} \qquad \frac{1}{2} \times \frac{15}{9} = \frac{15}{18}$$

Then $\frac{15}{18}$ is half-way between $\frac{7}{9}$ and $\frac{8}{9}$.
Use this method to find a number half-way between $\frac{3}{4}$ and $\frac{4}{5}$.

4. Use the above method for finding a number midway between:

a. $\frac{2}{5}$ and $\frac{3}{5}$ **b.** $\frac{13}{20}$ and $\frac{7}{10}$ **c.** $\frac{3}{20}$ and $\frac{2}{5}$ **d.** $\frac{1}{2}$ and $\frac{7}{12}$

5. Since each fraction has a decimal equivalent, we can use the method above with decimals.

a. Name a fractional number between 6.3 and 6.4 Answer: 6.35
b. Name a fractional number between 6.35 and 6.36. Answer: 6.355
Name a fractional number between 6.355 and 6.356.

6. Name two fractional numbers between 1.7 and 1.8.

7. Name three fractional numbers between 0.875 and 0.900.

Because you are always able to find a fractional number between any two fractional numbers, we say that the set of fractional numbers is a *dense set*. You can see that the set of whole numbers does not have this property, since you cannot find a whole number between any two consecutive whole numbers.

Part One

A. *Add:*

1. $\frac{1}{8} + \frac{3}{8}$
2. $\frac{5}{6} + \frac{5}{12}$
3. $\frac{3}{5} + \frac{9}{10} + \frac{1}{2}$
4. $8\frac{2}{3} + 4\frac{7}{12} + 7\frac{2}{3} + 8\frac{1}{2}$
5. $5\frac{4}{5} + 7\frac{3}{10} + 8\frac{1}{2} + 8\frac{7}{15}$
6. $7\frac{5}{6} + 3\frac{1}{2} + 9\frac{2}{3} + 4\frac{3}{4}$
7. 5916.43 + 16.57 + 104.39 + 6.825 + .342
8. 6.74 mi. + 34.01 mi. + 293.92 mi. + 6.85 mi. + 793.84 mi. + .0603 mi.

B. *Subtract:*

1. $\begin{array}{r} \frac{7}{8} \\ -\frac{1}{4} \\ \hline \end{array}$
2. $\begin{array}{r} 5\frac{5}{6} \\ -3\frac{2}{3} \\ \hline \end{array}$
3. $\begin{array}{r} 1\frac{1}{6} \\ -\frac{2}{3} \\ \hline \end{array}$
4. \$8 – \$1.75
5. 82.5 – 6.9
6. 42.008 – 16.5
7. 8.3 oz. – 6.315 oz.
8. 17.613 – 9.8
9. 6.07 – 5.985

C. *Multiply:*

1. $\frac{5}{8} \times \frac{4}{15}$
2. $\frac{5}{6} \times \frac{3}{10}$
3. $\frac{5}{8} \times \frac{2}{5}$
4. $\frac{3}{10} \times \frac{5}{9}$
5. $\frac{3}{4} \times \$16.60$
6. $\frac{7}{8} \times \$12.80$
7. 4.3 × .75
8. 8.6 × .355
9. .75 × \$4.15
10. .255 × \$16.50
11. 6.75 × \$5.50
12. .025 × \$160

D. *Divide:* Find the quotients to the nearest hundredth or to the nearest cent or in simplest form.

1. 451.3 ÷ .39
2. 37.993 ÷ 3.43
3. 1.875 ÷ .035
4. 4.7525 ÷ .095
5. 7.195 ÷ 3.128
6. 76.566 ÷ 4.96
7. \$68.24 ÷ 47
8. \$83.95 ÷ 16.7
9. $\frac{3}{8} \div \frac{9}{16}$
10. $\frac{2}{3} \div \frac{4}{9}$
11. $\frac{5}{9} \div \frac{3}{5}$
12. $6\frac{1}{4} \div \frac{5}{16}$

Part Two

A. Read each of the following questions carefully. Then write the letter corresponding to the correct answer after the numeral that identifies the question. There may be more than one correct answer.

1. Which of the following sets contains only natural numbers?
   a. {0,1,2} **b.** $\{\frac{1}{3},\frac{2}{3},1,\frac{4}{3}\}$ **c.** {5,7,9}
2. Which set contains only fractional numbers?
   a. $\{1,1\frac{1}{2},2,2\frac{1}{2}\}$ **b.** {1,3,5} **c.** {0,2,4,6}

3. Which set is made up of like fractions?

a. $\{\frac{1}{2},\frac{2}{4},\frac{3}{6}\}$ b. $\{\frac{1}{3},\frac{2}{3},\frac{4}{3}\}$ c. $\{\frac{3}{5},\frac{3}{7},\frac{3}{10}\}$

4. Which set is made up of equivalent fractions?

a. $\{\frac{1}{8},\frac{3}{8},\frac{5}{8}\}$ b. $\{\frac{5}{12},\frac{5}{6},\frac{5}{2}\}$ c. $\{\frac{3}{4},\frac{6}{8},\frac{9}{12}\}$

5. Which set contains only fractions in simplest form?

a. $\{\frac{1}{2},\frac{2}{4},\frac{4}{8}\}$ b. $\{\frac{3}{6},\frac{3}{9},\frac{3}{12}\}$ c. $\{\frac{2}{3},\frac{3}{4},\frac{4}{5}\}$

6. Which set contains only proper fractions?

a. $\{\frac{1}{3},\frac{2}{3},\frac{3}{3},\frac{4}{3}\}$ b. $\{\frac{1}{2},1\frac{1}{2},2\frac{1}{2}\}$ c. $\{\frac{1}{8},\frac{2}{8},\frac{3}{8}\}$

7. Which of the following is the multiplicative inverse for $2\frac{3}{4}$?

a. $4\frac{2}{3}$ b. $\frac{3}{8}$ c. $\frac{4}{11}$ d. $2\frac{4}{3}$

B. Find the number that n represents such that the second fraction will be equivalent to the first.

1. $\frac{3}{5}=\frac{n}{15}$ 3. $\frac{9}{16}=\frac{n}{64}$ 5. $\frac{1}{3}=\frac{n}{9}$ 7. $\frac{5}{8}=\frac{n}{48}$

2. $\frac{7}{9}=\frac{n}{45}$ 4. $\frac{3}{4}=\frac{n}{24}$ 6. $\frac{3}{5}=\frac{n}{25}$ 8. $\frac{7}{15}=\frac{n}{60}$

C. Rewrite each of the following as mixed numerals in simplest form or as numerals naming whole numbers.

1. $\frac{15}{3}$ 4. $\frac{18}{12}$ 7. $\frac{54}{12}$ 10. $\frac{108}{12}$

2. $\frac{24}{6}$ 5. $\frac{72}{12}$ 8. $\frac{55}{11}$ 11. $\frac{56}{13}$

3. $\frac{16}{6}$ 6. $\frac{88}{66}$ 9. $\frac{27}{4}$ 12. $\frac{48}{7}$

D. List the numerals 1 through 8 on a sheet of paper. After each write the answer to each of the questions about the Example at the right.

EXAMPLE

Divide:

$\frac{3}{8}\div\frac{3}{4}=?$

1. What is the dividend?
2. What is the divisor?
3. What is the numerator of the dividend?
4. What is the denominator of the divisor?
5. Rewrite the Example using the multiplicative inverse of the divisor.
6. What is the quotient?
7. Rewrite the Example using the decimal equivalents of dividend and divisor.
8. What is the quotient, when the Example is worked with decimal equivalents?

E. List the numerals 1 through 8 on a sheet of paper. After each numeral write the symbol >, <, or = to correctly complete the statement.

1. If two or more fractions have the same numerator, the one with the greatest denominator names a number __?__ the one named by any of the other fractions.
2. If a fractional number is divided by 1 the quotient is __?__ the dividend.
3. If a fractional number is multiplied by a second fractional number less than 1, the product is __?__ the first fractional number.
4. If two or more fractions have the same denominator, the one with the greatest numerator names a number __?__ any named by the other fractions.
5. If a fractional number is multiplied by 1 the product is __?__ the fractional number.
6. If a fractional number is divided by a number named by a mixed numeral, the quotient is __?__ the fractional number.
7. If a fractional number is multiplied by a number named by a mixed numeral, the product is __?__ the fractional number.
8. If both terms of a fraction are divided by the same number, the new fraction names a number __?__ the number named by the original fraction.

Part Three

1. In an orchard there are 10 rows of trees, with 100 trees in each row. The owner of the orchard says that 0.13 of the trees are too old and need to be replaced. How many young trees are needed?
2. A car traveled 141.9 miles on 8.6 gallons of gasoline. What was the average number of miles per gallon it traveled?
3. Mary is making a dress that will require $3\frac{1}{2}$ yards of cloth. She has $1\frac{1}{4}$ yards. How much more will she need?
4. A certain book company, in filling quantity orders of books, weighs the books instead of counting them. If a book weighs $1\frac{3}{8}$ pounds, and an order of 24 books is to be filled, how many pounds of books should be weighed out?
5. Distances at sea are measured in *nautical miles*. A nautical mile is about 1.15 land miles. The speed of a ship is expressed in *knots* which are *nautical miles per hour*. When a ship is traveling at 20 knots, how many miles per hour is it traveling?

CHAPTER FIVE

GEOMETRIC FIGURES

WORDS TO WATCH FOR

angle	*degree*	*midpoint*	*ray*
area	*diagonal*	*parallel*	*rectangle*
base	*diameter*	*parallelogram*	*region*
center	*edge*	*perimeter*	*right angle*
chord	*empty set*	*perpendicular*	*segment*
circle	*face*	*plane*	*trapezoid*
circumference	*formula*	*point*	*triangle*
clockwise	*geometry*	*polygon*	*equilateral*
collinear	*hexagon*	*prism*	*isosceles*
compass rose	*indirect measurement*	*protractor*	*right*
congruent	*intersection*	*quadrilateral*	*scalene*
counterclockwise	*line*	*ratio*	*vertex*

The branch of mathematics called *geometry* is a study of shape, size, and position. The word "geometry" is derived from two Greek words. These words are *gē*, which means the earth, and *metron*, which means measure. Actually, the study of geometry began when people noticed the various shapes of objects about them, such as the moon, spider webs, honeycombs, snowflakes, etc. As we study geometry, however, we will not necessarily think about such objects.

The geometry we will be studying is based on the work of a mathematician named Euclid, who lived more than 2000 years ago. Thus, the geometry that we will study is called *Euclidean* geometry.

In previous chapters we have dealt with sets of numbers. In geometry we are concerned with *sets of points*. The mathematician says a point has *neither length, width, nor thickness.*

1. A *segment* is a set of points consisting of two *endpoints* and including all the points between them. Draw a segment and label the two endpoints A and B.
2. Can you make a point with a pencil? We cannot define point, but we can represent it by a "dot," as we did in our earlier work with the number line.
3. According to the above description of a point, how many points would there be in a segment $\frac{1}{16}$ of an inch long? How many points would there be in a line segment 1 inch long?
4. The segment you drew for Exercise 1 is called $\overline{AB}$. The small bar over AB means "segment." How many members are in the set of points that make up $\overline{AB}$?
5. Any point on $\overline{AB}$ can be identified and named. Use a ruler to find the point halfway between A and B. Name that point M. It is called the *midpoint* of $\overline{AB}$.
6. Find the midpoint of $\overline{AM}$. Label that point R.
7. The set of points that makes up a segment is part of the set of points that makes up a *line*. A line has no endpoints, but continues indefinitely in each direction. What shows that $\overleftrightarrow{HJ}$ (Figure 1) has no endpoints? The small bar with the two arrowheads over HJ means "line."

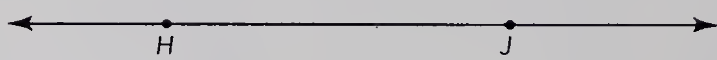


Figure 1

8. Just as two points determine a segment, so too, two points determine a line. However, a line can be named by only one letter, specifically a lower case letter. Draw a line, and name it line b. Locate three points on the line, and label them points X, Y, and Z. See Figure 2.

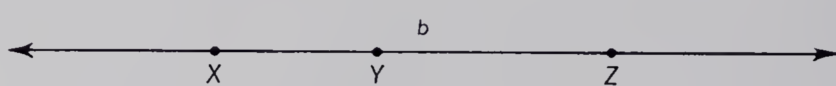


Figure 2

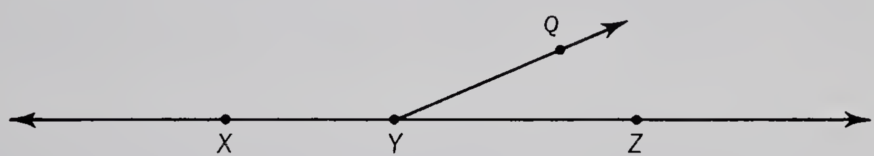
9. Is the set of points between and including Y and Z a segment? Explain your answer. Name two other segments in line b.
10. Since a line consists of a set of points, explain why a line has neither thickness nor width.

11. The set of points in line b that begins at Y and extends indefinitely in the Z direction forms a *ray*. In other words a "ray" has one endpoint and extends indefinitely in one direction. Therefore, we can also begin again at Y and extend indefinitely through X to make a ray. We call this $\overrightarrow{YX}$ or $\overleftarrow{XY}$. Note the bar over YX or XY has one arrowhead pointing in the X direction. How would you name the ray that begins at Y and extends through Z?

12. Does a ray have thickness or width?

13. We can say that the two rays $\overrightarrow{YZ}$ and $\overrightarrow{YX}$ together make up line b. Name a point M to the left of point X on line b. Use this point to help you name another pair of rays that make up line b.

14. We can say that $\overrightarrow{YZ}$ and $\overrightarrow{YX}$ make up line b, because a straight line goes through the three points, X, Y, and Z. We say X, Y, and Z are *collinear*. In your sketch of line b, label a point Q *not* on line b. Beginning at Y, draw a ray through Q. Do $\overrightarrow{YQ}$ and $\overrightarrow{YZ}$ make up a straight line? Explain.



15. Since Q, Y, and Z are not collinear, $\overrightarrow{YQ}$ and $\overrightarrow{YZ}$ do not make up a line. These 2 rays form an *angle*. An angle is formed by two rays that have the same endpoint. We will not say $\overrightarrow{YZ}$ and $\overrightarrow{YX}$ form an angle as they form a straight line, and we will not call a straight line an angle. Name another ray on line b that forms an angle with $\overrightarrow{YQ}$.

16. The angle formed by $\overrightarrow{YQ}$ and $\overrightarrow{YZ}$ is called "angle QYZ" or "angle ZYQ." We use the symbol, $\angle$, to stand for the word "angle." The endpoint that $\overrightarrow{YQ}$ and $\overrightarrow{YZ}$ have in common, Y, is called the *vertex* of the angle. Name the angle formed by $\overrightarrow{YQ}$ and $\overrightarrow{YZ}$ in two ways.

17. Draw an angle and label the vertex F. In each of the two rays that make up the angle find a point and label them respectively G and H.

18. Is point F a member of the set of points that make up $\overrightarrow{FG}$?

19. Is point F a member of the set of points that make up $\overrightarrow{FH}$?

20. Is there any other point that is a member of both sets of points that make up $\overrightarrow{FG}$ and $\overrightarrow{FH}$?

Lines and Segments

All geometric figures are made up of a set of points. A segment is made up of two endpoints and all the points between them.

1. For a point to lie "between" two other points, all three points must be collinear (lie on the same line). Which point is between D and E in Figure 1?

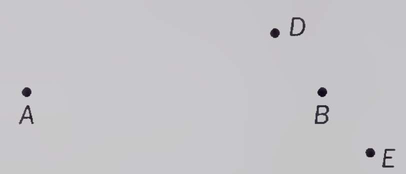


Figure 1

2. Which point in Figure 1 is between A and C?

3. On your paper represent a set of points in a similar pattern as in Figure 1. Draw $\overline{AC}$ and $\overline{DE}$. What point do $\overline{AC}$ and $\overline{DE}$ have in common?

4. Since B is common to both $\overline{AC}$ and $\overline{DE}$, we say that B is a member of *both* sets of points. The *intersection* of two sets is the set containing all the elements that are members of both sets. We will use the symbol "$\cap$" to mean "the intersection of." Is the following statement true? $\overline{AC} \cap \overline{DE} = \{B\}$

5. Write the statement to show the intersection of $\overline{PQ}$ and $\overline{RS}$ in Figure 2.

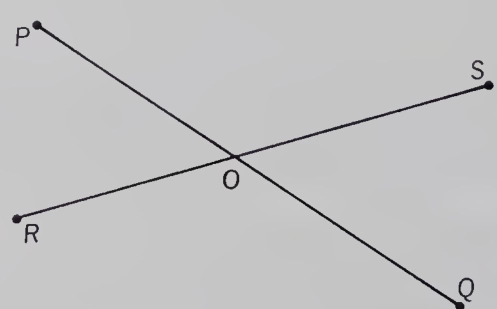


Figure 2

6. In Figure 3, does B belong to $\overline{AE}$? Explain your answer.

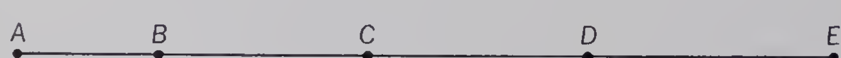


Figure 3

7. What other points that are labeled belong to $\overline{AE}$?

8. Is this statement true for Figure 3?

$$\overline{AD} \cap \overline{CE} = \overline{CD}$$

9. Write two other similar true statements about Figure 3.

10. In Figure 4 there are how many intersecting segments? Complete these statements about Figure 4.

   a. $\overline{AB} \cap \overline{BC} = ?$
   b. $\overline{AC} \cap \overline{BC} = ?$
   c. $\overline{AB} \cap \overline{AC} = ?$

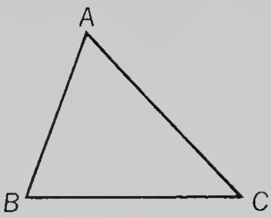


Figure 4

11. Can two segments have more than one point in common? Examine Figure 3 before you answer.

12. Remember! The line in Figure 5 is named $\overleftrightarrow{AB}$ or $\overleftrightarrow{BA}$, as you choose. Does it have any endpoints? Explain your answer.

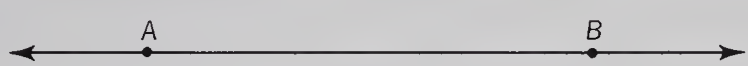


Figure 5

13. Draw $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ such that $\overleftrightarrow{RS}$ intersects $\overleftrightarrow{PQ}$. Label the point of intersection O. Can the lines intersect at any other point? Explain your answer.

14. Complete this statement: $\overleftrightarrow{PQ} \cap \overleftrightarrow{RS} = ?$

15. Complete this statement: Two different lines can have $\underline{\ ?\ }$ point(s) in common.

16. Complete this statement: A line is determined by $\underline{\ ?\ }$ point(s).

17. Draw $\overline{AB}$ and locate C between A and B. Can you locate D between A and C? If so, indicate its location on your drawing.

18. Is there a point E between A and D?

19. Is there a point F between A and E?

20. Could you continue this indefinitely? If so, what can you say of the number of points between A and B?

21. Can you locate the point *next* to A on $\overline{AB}$? Explain your answer.

22. Draw $\overline{PQ}$. What are its endpoints?

23. Think of the segment made up of all the points between but *not* including P and Q. This is called an *open segment*. It is named $\overset{\circ\!-\!\circ}{PQ}$. Write a definition for an open segment.

1. One unit of measure for angles is the *degree,* indicated by a small circle: °. In Figure 1 below, $\angle ABC$ is a *right angle* measuring 90°. Because $\overrightarrow{BC}$ and $\overrightarrow{BA}$ form a right angle, we say that $\overrightarrow{BC}$ and $\overrightarrow{BA}$ are *perpendicular* to each other. Figure 2 below shows that a circle measures 360°. What fraction of 360° is the measure of a right angle?

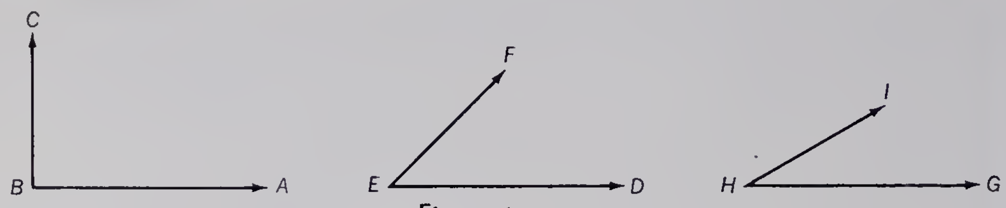


Figure 1

2. The measure of $\angle DEF$, Figure 1, is equal to one-half the measure of $\angle ABC$. Using the shorthand "$m\angle A$" to mean "the measure of angle A," this sentence could be written as: $m\angle DEF = \frac{1}{2}m\angle ABC$ What is the measure of $\angle DEF$?

3. The measure of $\angle GHI$ is equal to one-third the measure of $\angle ABC$ ($m\angle GHI = \frac{1}{3}m\angle ABC$). What is the measure of $\angle GHI$?

4. What fraction of 360° is $m\angle GHI$?

5. What is the measure of $\angle AOB$ in Figure 2?

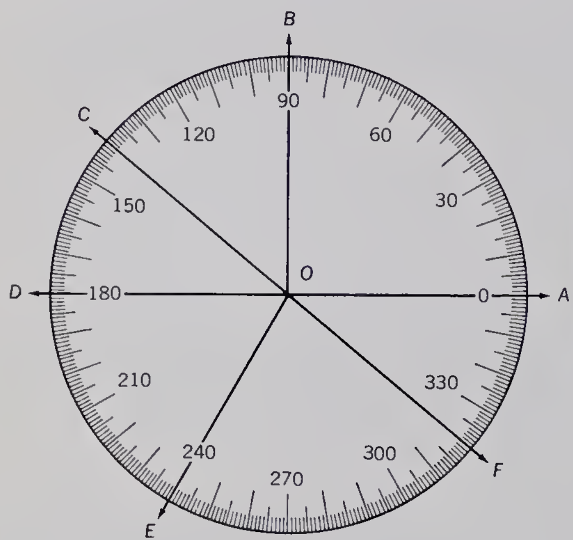


Figure 2

Figure 3

6. What kind of angle is $\angle AOB$?

7. Remember! In naming an angle, the letter at the vertex of the angle is written between the other two letters. Name all angles in Figure 2 with each of the following measures.

a. 80° **b.** 60° **c.** 40°

8. Figure 3 on page 168 shows a mariner's compass, commonly called a compass rose. What is the measure of the angle formed by the rays that go through the following pairs of points? (The vertex of each of these angles is the "center" of the compass rose.)

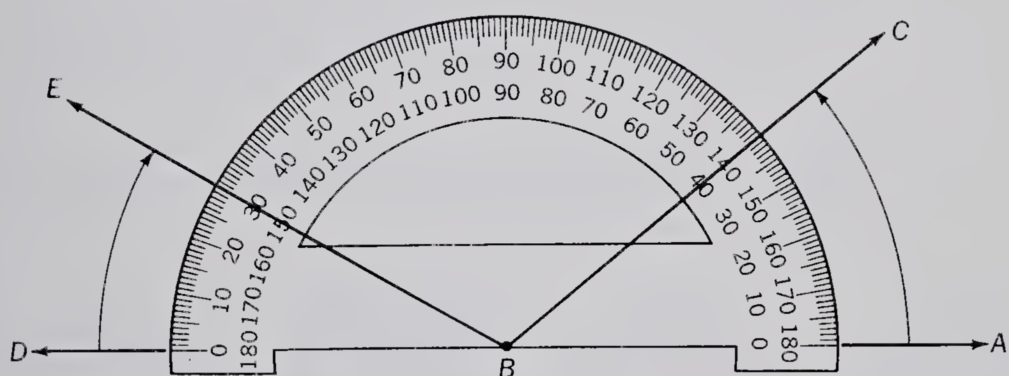
a. E and N b. W and SW c. S and SSW d. SE and N

9. What is the measure (less than 180°) of the angle determined by the hands of a clock at the following times:

a. 3 o'clock b. 1 o'clock c. 7 o'clock d. 9 o'clock

10. An angle measuring less than 90° but greater than 0° is called an *acute* angle. An angle measuring more than 90° and less than 180° is called an *obtuse* angle. In Exercise 9 when do the hands of the clock determine an acute angle? an obtuse angle?

11. The number of degrees in an angle is measured by an instrument called the *protractor*. (See the Figure below.) Notice the numerals on the protractor. We use the numerals on the *inner* part when measuring angles placed like $\angle ABC$. To measure an angle (using the inner scale) place the vertex of the angle at the vertex of the protractor. Be sure that a ray passes through the 0 mark on the protractor, as $\overrightarrow{BA}$ does. Read the measure of the angle by looking at the other ray, such as $\overrightarrow{BC}$. What is the measure of $\angle ABC$? In using the inner scale, we are measuring the angle in a *counterclockwise* direction.



12. Why would you use the outer scale to measure $\angle DBE$?

13. The outer scale is used to measure angles in a "clockwise" direction. What is the measure of $\angle DBE$? Notice how $\angle DBE$ is placed in order to use the outer scale.

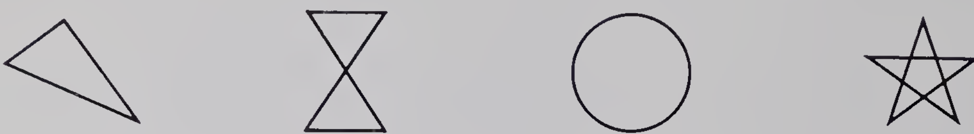
14. What is the measure of $\angle EBC$?

15. Using a protractor, draw angles with the following measures first by using the inner scale and then the outer scale.

a. 30° b. 60° c. 45° d. 90° e. 120° f. 135° g. 150°

A *simple polygon* is a closed broken line such that the segments meet only at their endpoints. Also, no more than two segments intersect at a vertex. The adjective "closed" means that if you were to draw a polygon, your starting point would also be your endpoint. That is, you can draw one without ever lifting your pencil from the paper.

Which of the following are simple polygons?



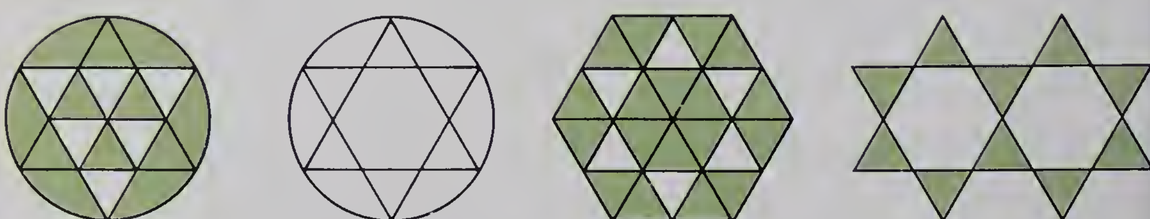
a. c. e. g.

b. d. f. h.

Note: The adjective "simple" is important, because Figure c above is a polygon but it is not a simple polygon. The segments of a polygon can intersect at places other than at their vertices. However, simple polygons meet *only* at their endpoints. Since our discussion will only be concerned with simple polygons, it will not be necessary to use the adjective "simple."

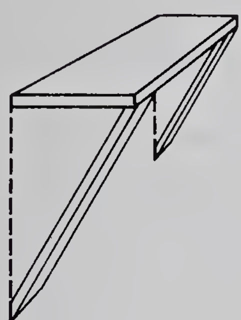
A *triangle* is a three-sided polygon. Which of the above is a triangle? Have you ever noticed how widely triangles are used in our everyday lives? In general, you will find them used for these important purposes:

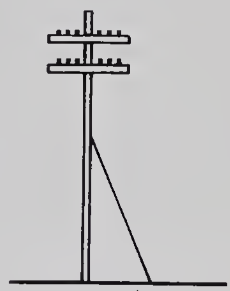
(a) *In ornamentation and design.* Triangles are commonly the basis for designs in tile, tapestry, and linoleum, for example. They are also utilized for ornamental purposes in architecture.



(b) *For indirect measurements of heights and distances.* An indirect measurement is necessary when it is impossible to reach a point directly to make the measurement. Thus the distance from a ship to a point on the shore, the height of a mountain, or the distance to the moon must all be measured indirectly. Triangles are utilized for all these measurements. See the Figure at the top of page 171.

(c) *As a unit in rigid construction.* The shape of a triangle may not be changed without pulling it apart at one of its vertices or breaking one of its sides. This is why we say that a triangle is a rigid figure. If you watch a house or a bridge being built, or a telegraph pole being braced, or a sagging screen straightened out by running a wire diagonally down to the sagging corner, you will see an example of the use of a triangle in rigid construction.

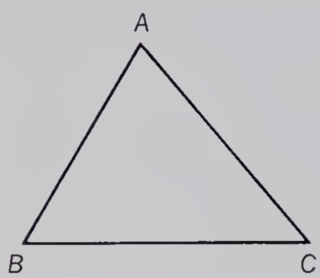


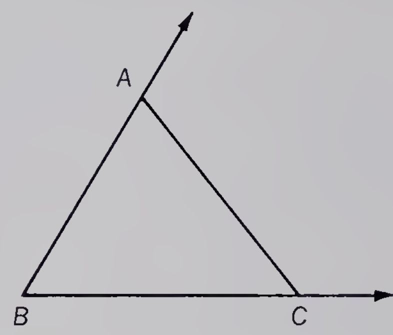


Because triangles are so important in our world, it is worth-while to learn more about their characteristics.

1. Find a house or other building with timber construction, and make a sketch of the way in which triangles are used to make the construction rigid.
2. Find and sketch some examples of triangles being used to secure rigid construction through bracing or as a prop.
3. Examine a bridge, or a picture of a bridge, to see how the triangle enters into the construction. Make a sketch to show where the triangle is used.
4. Examine some pictures of Greek or Roman architecture, and see if you can find cases where the use of the triangle is for ornamental purposes, rather than to secure rigid construction.
5. Are triangles used for ornamental purposes on modern buildings? See if you can find and sketch an example.
6. Find an example of each of the three uses of triangles in the Sketches on pages 170 and 171.

Each pair of sides of the triangle below *determines* an angle. Since the sides of the triangle are line segments, not rays, there are actually no angles in the triangle. Using B as an endpoint, however, rays can be drawn through points A and C. The angle with its vertex at B is determined by the triangle. Show how you can draw rays to form angles with vertices at C and A.





The three angles determined by the sides of a triangle are commonly called the *angles of the triangle.* Thus a triangle is made up of three sides (segments) and three angles.

With a ruler, compass, and protractor you will find it easy to draw a triangle, given three parts, one or more of which is a side. In regard to the sides (segments), we will use AB to mean "the measure of $\overline{AB}$." Similarly, BC means "the measure of $\overline{BC}$."

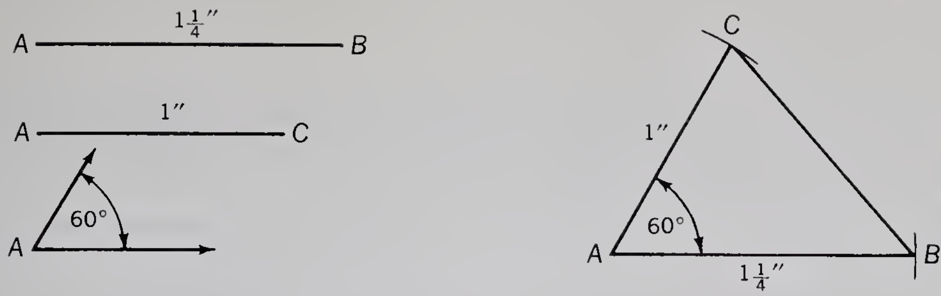
Given Two Sides and the Included Angle

The following Example illustrates how to construct a triangle given the measures of two sides and the included angle. Refer to the Figure at the top of page 173 as you study the steps.

EXAMPLE

Given: $AB = 1\frac{1}{4}$ inches; $AC = 1$ inch; $m\angle A = 60°$

1. Draw a segment somewhat longer than the longest given side.
2. Set your compass at $1\frac{1}{4}$ inches. Measure off this distance with the compass on the segment you drew. Label the endpoints A and B.
3. At point A construct an angle whose measure equals that of the given angle A (60°). Extend this side somewhat more than 1 inch.
4. Using your compass, mark off AC (in this case, 1 inch), on the second side of angle A. Label the endpoint C.
5. Draw $\overline{BC}$.



Construct triangles with sides and angles measuring as follows:

	AB	AC	$m\angle A$		AB	AC	$m\angle A$
1.	3″	3″	40°	**4.**	$2\frac{1}{2}''$	$2\frac{1}{2}''$	60°
2.	2″	3″	65°	**5.**	2″	$1\frac{1}{2}''$	90°
3.	$1\frac{1}{2}''$	$2\frac{1}{2}''$	30°	**6.**	2″	$2\frac{3}{4}''$	45°

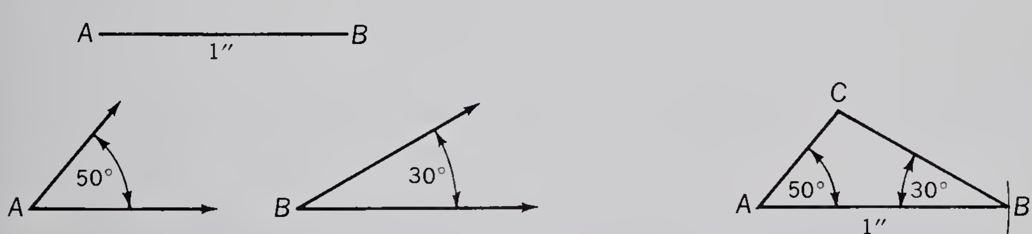
Given Two Angles and the Included Side

The following Example illustrates how to construct a triangle given the measures of two angles and the included side.

EXAMPLE

Given: $AB = 1$ inch; $m\angle A = 50°$; $m\angle B = 30°$

1. Draw a segment somewhat longer than the given side.
2. Set your compass at 1 inch. Measure off this distance on the segment you drew. Label the endpoints A and B.
3. At point A, construct an angle equal to the measure of angle A.
4. At point B, construct an angle equal to the measure of angle B.
5. Extend the sides (rays) of angles A and B until they intersect. Label the point of intersection C.



Construct triangles with sides and angles measuring as follows:

	AB	$m\angle A$	$m\angle B$		AB	$m\angle A$	$m\angle B$
1.	2 in.	60°	80°	**4.**	4 in.	60°	60°
2.	$2\frac{1}{2}$ in.	45°	45°	**5.**	3 in.	48°	72°
3.	$3\frac{1}{2}$ in.	90°	35°	**6.**	1 in.	90°	60°

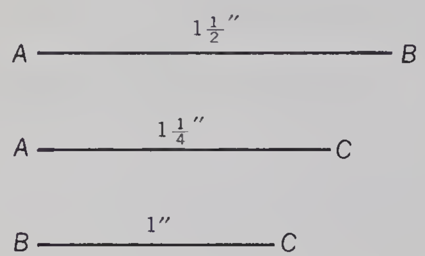
Given the Three Sides

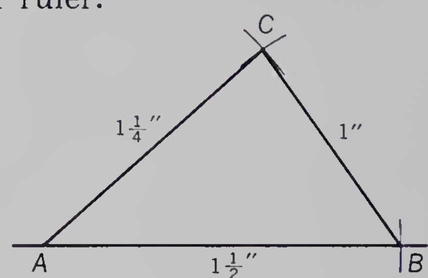
The following Example illustrates how to construct a triangle given the measures of the three sides.

EXAMPLE

Given: $AB = 1\frac{1}{2}$ inches; $AC = 1\frac{1}{4}$ inches; $BC = 1$ inch

1. Draw a segment somewhat longer than the longest given side.
2. Set your compass at $1\frac{1}{2}$ inches. Measure off this distance on the segment you drew. Label the endpoints A and B.
3. Set your compass at $1\frac{1}{4}$ inches. This is side $\overline{AC}$. Using point A as an endpoint, draw an arc above $\overline{AB}$.
4. Set your compass at 1 inch. This is side $\overline{BC}$. Using point B as an endpoint, draw an arc that will intersect the first arc. Label the point of intersection C.
5. Draw the sides $\overline{AC}$ and $\overline{BC}$ with your ruler.





Use the above steps and construct triangles having sides measuring as follows:

	AB	BC	AC
1.	$1\frac{1}{2}$ in.	2 in.	$2\frac{1}{2}$ in.
2.	$3\frac{1}{2}$ in.	2 in.	$2\frac{1}{4}$ in.
3.	4 in.	4 in.	4 in.
4.	4 in.	3 in.	2 in.
5.	$2\frac{1}{2}$ in.	$3\frac{1}{2}$ in.	$3\frac{1}{2}$ in.

Now construct the triangles with the following measures.

	AB	AC	BC	$m\angle A$	$m\angle B$	$m\angle C$
6.	$1\frac{1}{2}''$	$2''$	—	90°	—	—
7.	$4''$	—	—	30°	40°	—
8.	$2\frac{1}{2}''$	—	$2\frac{1}{2}''$	—	60°	—
9.	$3''$	$4''$	$5''$	—	—	—
10.	$2''$	$1\frac{1}{2}''$	—	110°	—	—

Using your protractor, measure each angle in the ten triangles you drew in Exercises 1–10, on page 174. Prepare a table like this:

Triangle No.	$m\angle A$	$m\angle B$	$m\angle C$	*Sum of the Measures of the Angles*
1	?	?	?	?
2	?	?	?	?
etc.				

1. Is the sum of the measures of the angles approximately 180° in each triangle? If the sum of the measures is not *exactly* 180°, it is probably because your measurements are not precise. Can you measure precisely 1°?

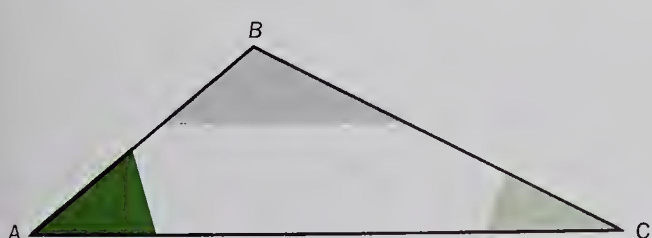
2. If all three sides of a triangle have the same measure, then it is an *equilateral* triangle. (The prefix "equi-" means equal and "lateral" refers to sides.) Explain why the angles of an equilateral triangle each measure 60°.

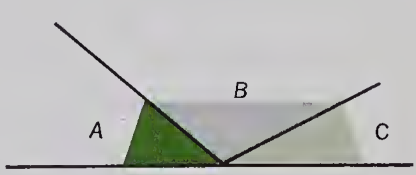
3. An *isosceles* triangle is a triangle that has two sides with the same measure. If two angles of an isosceles triangle each measure 45°, what is the measure of the third angle?

4. A *scalene* triangle is a triangle each side of which has a different measure. If two angles of a scalene triangle measure 50° and 85°, what is the measure of the third angle?

5. A *right* triangle is a triangle that contains a right angle (90°). Therefore, the sides of the triangle that determine the right angle are perpendicular to each other. If an angle of a right triangle measures 40°, what is the measure of the third angle? Can a right triangle be isosceles? equilateral?

6. Here is another way to show that the sum of the measures of the angles of a triangle is 180°. Cut out any one of the triangles you have drawn, and "cut off" the angles. (See the Figure below.) When you fit them together their two "outer" sides will form a straight line. How many degrees does the straight line represent?





7. Explain this fact: A triangle can have no more than one right angle.

8. Explain this fact: A triangle can have only one obtuse angle.

The triangles ABC and DEF in Figure 1 below have the same size and shape. If you would cut out these two triangles and place one upon the other, you would find that the sides and angles matched exactly. We say that one would fit exactly on the other.

Note: We will use the symbol $\triangle$ to represent a triangle.

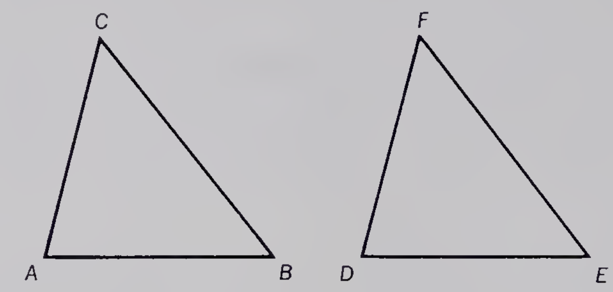


Figure 1

1. Which side of $\triangle ABC$ would match side $\overline{DE}$ of $\triangle DEF$?
2. We call sides $\overline{AB}$ and $\overline{DE}$ *corresponding parts* because in each case the relationship of that part to the whole triangle is the same. Which side of $\triangle DEF$ corresponds to side $\overline{AC}$ of $\triangle ABC$?
3. Which angle of $\triangle ABC$ corresponds to $\angle D$?
4. Which angle of $\triangle DEF$ corresponds to $\angle C$?
5. In triangles RTW and STW, Figure 2, what side corresponds to side $\overline{RW}$?

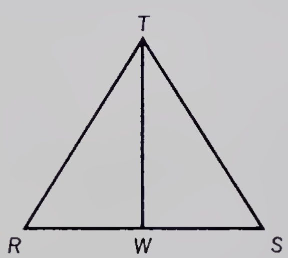


Figure 2

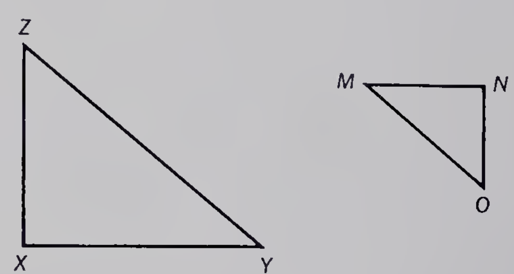


Figure 3

6. Which angle corresponds to $\angle R$?
7. Which side of $\triangle STW$ corresponds to side $\overline{WT}$ in $\triangle RTW$?
8. It is not necessary that figures be the same size in order to have corresponding parts. Triangles MNO and XYZ, Figure 3, are the same shape. Which side of $\triangle XYZ$ corresponds to side $\overline{MN}$?
9. Which angle of $\triangle MNO$ corresponds to $\angle X$?
10. Which side corresponds to side $\overline{YZ}$?

Suppose you wished to find the measure from point A to point C (Figure 1). If you measure the distance from A to B, and also angles BAC and ABC, you can find the measure from A to C, as shown in Figure 1. Since you know the measure of one side and that of two angles, you can construct the triangle on dry land, and measure the length from A to D, which is the same as from A to C. This is an illustration of the use of *indirect measurement,* which was briefly discussed on page 170.

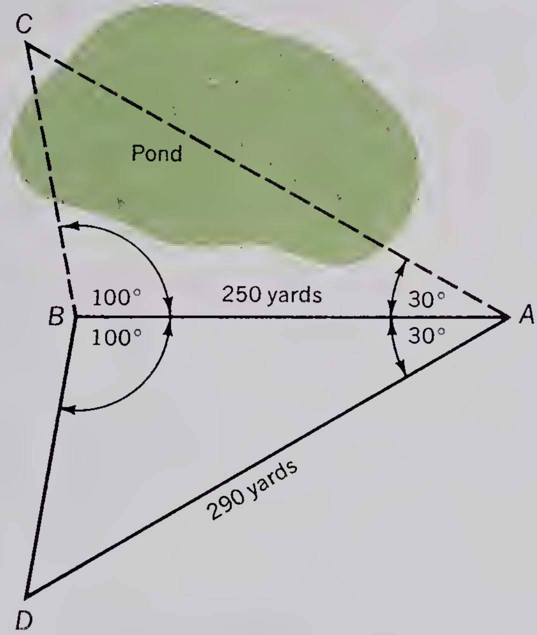


Figure 1

When two figures have the same shape and size they are *congruent.* In Figure 1 we know that $\triangle ABD$ is congruent to $\triangle ABC$ because:

Two triangles are congruent if the measures of two angles and the included side of one triangle are equal in measure to the corresponding parts of the other.

1. You could construct a triangle congruent to $\triangle ABC$ (Figure 2), using two angles and the included side. You could also use two sides and the included angle. Using the measures of sides $\overline{AB}$, $\overline{AC}$, and angle A, construct the triangle. Is the one you constructed congruent to the one in Figure 2? How can you tell?

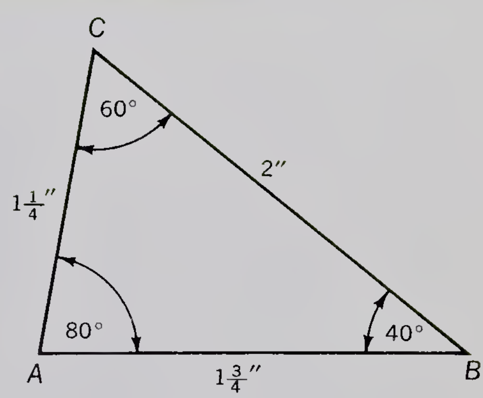


Figure 2

2. You have just illustrated this fact:

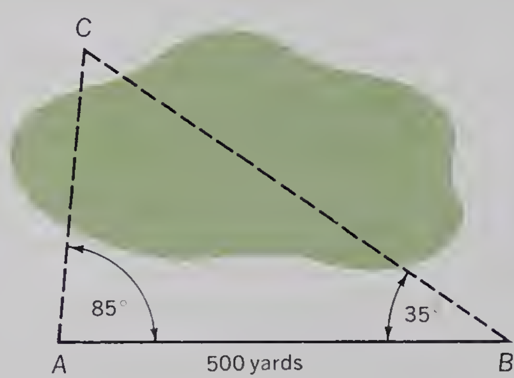
> Two triangles are congruent if the measures of two sides and the included angle of one triangle are equal in measure to the corresponding parts of the other.

Does the triangle that you drew have two sides and the included angle equal in measure to the corresponding parts in Figure 2?

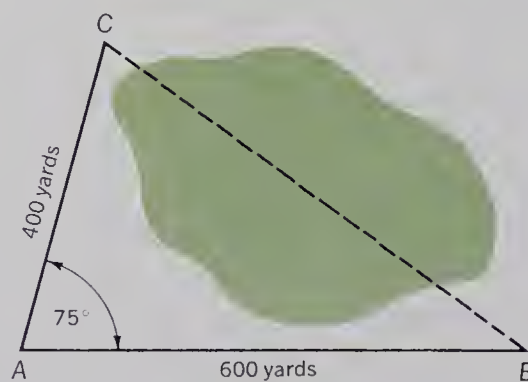
3. You can also draw a triangle congruent to $\triangle ABC$ (Figure 2) by making use of this fact:

> Two triangles are congruent if the measures of the three sides of one triangle are equal to the measures of the three sides of the other.

4. Construct a triangle congruent to $\triangle ABC$ without measuring the angles.
5. If you wished to measure side $\overline{BC}$ in the triangle below, which of the three construction methods would you use?

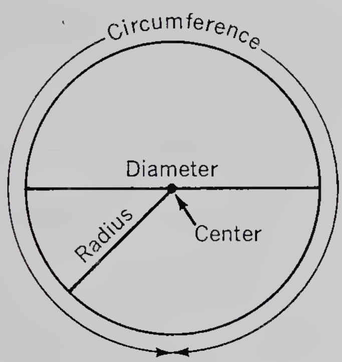


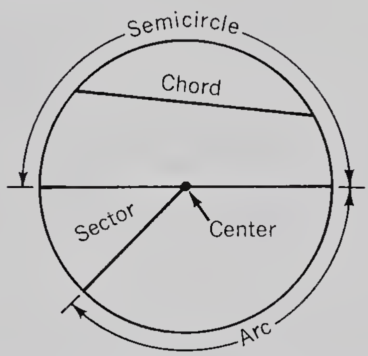
6. Which of the three construction methods would you use to measure side $\overline{BC}$ in the triangle below?



How well do you remember the vocabulary relating to circles? List the numerals 1 through 10. After each numeral write the word from the right-hand column that is described by the statement having that numeral.

1. Half of a circle	arc
2. A segment whose endpoints are on the circle	area
3. The plural of radius	center
4. The point where two diameters cross	chord
5. A chord through the center of the circle	circumference
6. A segment joining the center of the circle and a point on the circle	degree
7. A part of the circle	diameter
8. The circumference divided by the measure of the diameter	pi
9. A part of the circular region bounded by the circle and two radii	radii
10. The measure of the length of the circle	radius
	sector
	segment
	semicircle
	vertex





Study the Figures above, and see how many of the questions you were unable to answer correctly. Use the Glossary in the back of the book to find any definitions you were not certain of. Then try the test again and see if you can do better.

The geometric figure that we call a circle is a set of points. Each point on the circle is equally distant from a point in the interior of the circle, which, of course, is the center of the circle. A diameter is also a set of points. No matter how many diameters you may draw in a given circle, they will all have the same measure. A radius is still another set of points. Here again, regardless how many radii you may draw in a given circle, they will all have the same measure. The distance around a circle is the circumference. Therefore, it is not a set of points; it is a measure.

1. Using a ruler, measure the diameters of several "circular" shaped objects, for example, the tops of cans.
2. Using some string and a ruler, find the circumference of the same object. Make sure you record the measure of the diameter and the circumference for each object.
3. Divide the circumference by the measure of the diameter for each object. Examine each of your answers.

Depending on how accurate your measurements were, each of your answers to Exercise 3 above should be approximately the same. This number that is obtained by dividing the circumference by the measure of a diameter of a circle is represented by the Greek letter π. It is approximately equal to the fraction $3\frac{1}{7}$ or the decimal 3.14 (rounded to the nearest hundredth). The value of π cannot be expressed precisely, but it has been calculated to very many decimal places.

Use either $3\frac{1}{7}$ or 3.14, whichever is more convenient, whenever you are solving problems.

We can express π as follows:

$$\pi = \text{the circumference} \div \text{the measure of a diameter}$$

or

$$\pi = \frac{\text{the circumference}}{\text{the measure of a diameter}}$$

If we let c represent the circumference and d represent the measure of a diameter, the above is simplified:

$$\pi = c \div d \qquad \text{or} \qquad \pi = \frac{c}{d}$$

We call the fraction $\frac{c}{d}$ the *ratio* of the circumference to the measure of a diameter.

Keeping in mind that

$$\text{factor} \times \text{factor} = \text{product}$$

and

$$\text{product} \div \text{factor} = \text{unknown factor}$$

the relationship $\pi = \frac{c}{d}$ becomes

$$\pi \times d = c \qquad \text{or} \qquad c = \pi \times d$$

This can further be simplified to

$$c = \pi d$$

1. Use the relationship, $c = \pi d$, which we call a *formula*, to find the circumference of a circle with a diameter measuring 3 feet.
2. The free throw circle on a basketball court has a radius that measures 6 feet. What is the distance around the circle?
3. The circle at the center of a basketball court has a 2-foot radius. What is the circumference of this circle?
4. The basket ring has a diameter that measures 18 inches. What is the distance around the ring?
5. A standard basketball has a circumference of 30 inches. What is the measure of a radius of a standard basketball?
6. Find the measure of a diameter of a tree whose circumference is 7 feet 10 inches.
7. Is your answer to Exercise 6 accurate or approximate? Explain.
8. The standard size bicycle wheel has a diameter that measures 26 inches. What is the circumference of the wheel?
9. Jimmy measured a diameter of his bicycle wheel with tire and found it to be 29 inches. What is the circumference of the wheel with tire?
10. The large sprocket wheel on Jimmy's bike measures 8 inches in diameter. What is the circumference of the sprocket wheel?
11. The top of the wastebasket in Jimmy's room has a diameter that measures 16 inches. The bottom has a diameter that measures 12 inches. What is the difference in the circumferences of the top and bottom?
12. The distance around the world at the equator is about 25,000 miles. What is the distance through the center (diameter) of the earth to the nearest thousand miles?
13. A diameter of the front wheel on Mr. Foote's car measures 28 inches. How far does the car move when the wheel has made 10 revolutions?
14. How many times does the wheel turn when the car travels one mile?
15. The measure of a semicircle is $\frac{11}{35}$ inches. (Remember! A semicircle is half a circle.) What is the measure of a radius of the semicircle?
16. Find the circumference of a circle if a diameter measures:
    a. 2 inches **b.** 4 inches **c.** 8 inches **d.** 16 inches
17. What happens to the circumference of a circle when we double the measure of a diameter?

A. *Addition:*

1. $\begin{array}{r}\frac{5}{8}\\ \frac{1}{2}\\ \frac{3}{4}\\ \frac{5}{16}\\ \hline\end{array}$

2. $\begin{array}{r}\frac{7}{9}\\ \frac{2}{3}\\ \frac{5}{12}\\ \frac{17}{18}\\ \hline\end{array}$

3. $\begin{array}{r}\frac{2}{5}\\ \frac{7}{10}\\ \frac{1}{2}\\ \frac{3}{4}\\ \hline\end{array}$

4. $\begin{array}{r}\frac{7}{12}\\ \frac{2}{3}\\ \frac{5}{6}\\ \frac{1}{4}\\ \hline\end{array}$

5. $\begin{array}{r}8\frac{1}{6}\\ 7\frac{1}{2}\\ 15\frac{2}{3}\\ 9\frac{1}{4}\\ \hline\end{array}$

6. $\begin{array}{r}16\frac{1}{2}\\ 11\frac{2}{5}\\ 17\frac{3}{10}\\ 5\frac{11}{15}\\ \hline\end{array}$

7. $\begin{array}{r}4\frac{1}{8}\\ 11\frac{5}{12}\\ 16\frac{2}{3}\\ 5\frac{11}{16}\\ \hline\end{array}$

8. $\begin{array}{r}14\frac{2}{3}\\ 16\frac{1}{3}\\ 3\frac{5}{6}\\ 17\frac{10}{24}\\ \hline\end{array}$

9. $\begin{array}{r}13\frac{5}{8}\\ 9\frac{1}{4}\\ 23\frac{2}{3}\\ 5\frac{5}{6}\\ \hline\end{array}$

10. $\begin{array}{r}9\frac{5}{9}\\ 3\frac{11}{12}\\ 26\frac{2}{3}\\ 48\frac{3}{4}\\ \hline\end{array}$

11. $\begin{array}{r}5\frac{2}{3}\\ 7\frac{1}{2}\\ 26\frac{5}{6}\\ 13\frac{7}{12}\\ \hline\end{array}$

12. $\begin{array}{r}35\frac{3}{8}\\ 16\frac{1}{5}\\ 8\frac{3}{4}\\ 25\frac{7}{10}\\ \hline\end{array}$

B. *Subtraction:*

1. $\begin{array}{r}45\frac{1}{8}\\ 17\frac{1}{16}\\ \hline\end{array}$

2. $\begin{array}{r}17\frac{2}{5}\\ 13\\ \hline\end{array}$

3. $\begin{array}{r}26\frac{1}{3}\\ 15\frac{5}{9}\\ \hline\end{array}$

4. $\begin{array}{r}42\frac{1}{2}\\ 27\frac{1}{8}\\ \hline\end{array}$

5. $\begin{array}{r}86\frac{5}{8}\\ 59\frac{15}{16}\\ \hline\end{array}$

6. $\begin{array}{r}49\frac{1}{16}\\ 25\frac{1}{2}\\ \hline\end{array}$

7. $\begin{array}{r}35\\ 6\frac{5}{8}\\ \hline\end{array}$

8. $\begin{array}{r}72\frac{1}{2}\\ 18\frac{3}{4}\\ \hline\end{array}$

9. $\begin{array}{r}16\frac{2}{3}\\ 8\frac{5}{12}\\ \hline\end{array}$

10. $\begin{array}{r}37\frac{1}{5}\\ 9\frac{1}{4}\\ \hline\end{array}$

C. *Multiplication:*

1. $\begin{array}{r}8\\ 1\frac{1}{4}\\ \hline\end{array}$

2. $\begin{array}{r}16\\ 4\frac{7}{8}\\ \hline\end{array}$

3. $\begin{array}{r}6\frac{5}{6}\\ 8\frac{1}{2}\\ \hline\end{array}$

4. $\begin{array}{r}13\frac{2}{7}\\ 9\frac{4}{5}\\ \hline\end{array}$

5. $\begin{array}{r}36\\ 12\frac{5}{8}\\ \hline\end{array}$

6. $\begin{array}{r}5\frac{5}{6}\\ 3\frac{1}{5}\\ \hline\end{array}$

7. $\begin{array}{r}18\\ 6\frac{2}{3}\\ \hline\end{array}$

8. $\begin{array}{r}6\frac{2}{3}\\ 8\frac{1}{2}\\ \hline\end{array}$

9. $\begin{array}{r}17\frac{5}{9}\\ 16\frac{3}{5}\\ \hline\end{array}$

10. $\begin{array}{r}35\\ 7\frac{3}{7}\\ \hline\end{array}$

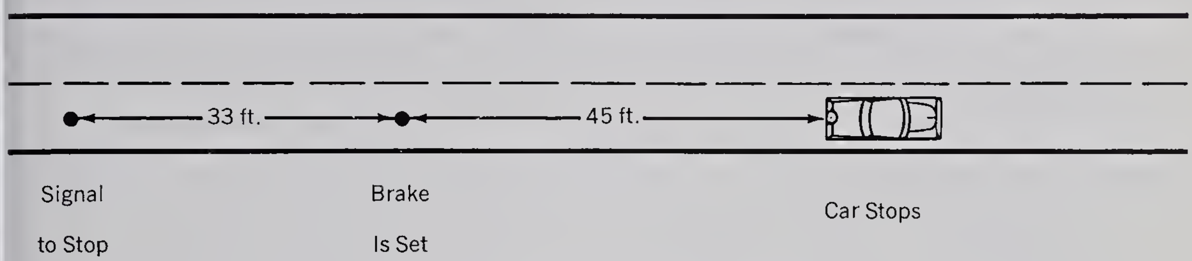
D. *Division:*

1. $\frac{1}{6} \div 3$
2. $\frac{5}{6} \div 15$
3. $5\frac{1}{2} \div \frac{1}{7}$
4. $4\frac{1}{5} \div \frac{7}{10}$
5. $7\frac{2}{3} \div 1\frac{7}{8}$
6. $6\frac{2}{3} \div 7\frac{4}{5}$
7. $8 \div \frac{4}{9}$
8. $32 \div \frac{8}{15}$
9. $6\frac{3}{5} \div 17\frac{2}{3}$

If you need more practice, turn to the Exercises on page 461 and following. If not, you may work in the Experts' Corner on the following page.

How Quickly Can You Stop a Car?

The car used for driver training at the Edison High School was equipped to measure the distance it took to stop after a signal was given. When Jim was driving, for example, Clarence would give the signal to stop, and a cartridge would fire making a red paint spot on the pavement. When Jim's foot touched the brake, another cartridge would fire, making another spot. When the car was stopped, the distances could be measured.



STOPPING OF CAR AT 30 MPH.

You can see from the diagram that the total stopping distance includes two distances: (a) the distance the car traveled before Jim's foot touched the pedal, and (b) the distance the car traveled after the brakes were set.

1. The time that elapses after you see the danger until you set the brakes is called the *reaction* time. Tests show that this is about $\frac{3}{4}$ of a second for the average person. If a car is traveling 44 feet per second, how far will it travel during the reaction time?

2. How many feet per second is a car traveling if it is going 60 miles per hour?

3. With an average reaction time, how far will a car travel between the time danger is sighted and the brakes are set if the car is traveling at 60 miles per hour?

4. Clarence and Jim found that after the brakes are set, the "braking distance" on dry pavement depends on how fast the car is traveling; the distance increased with the *square* of the speed of the car, that is:

If the speed is doubled, the braking distance is increased 4 times (2^2). If the speed is tripled, the braking distance is increased 9 times (3^2). The braking distance of the average car, at 10 miles per hour, is 5 feet. What is the braking distance at 20 miles per hour?

5. Using the figures from Exercise 4, what is the braking distance at:

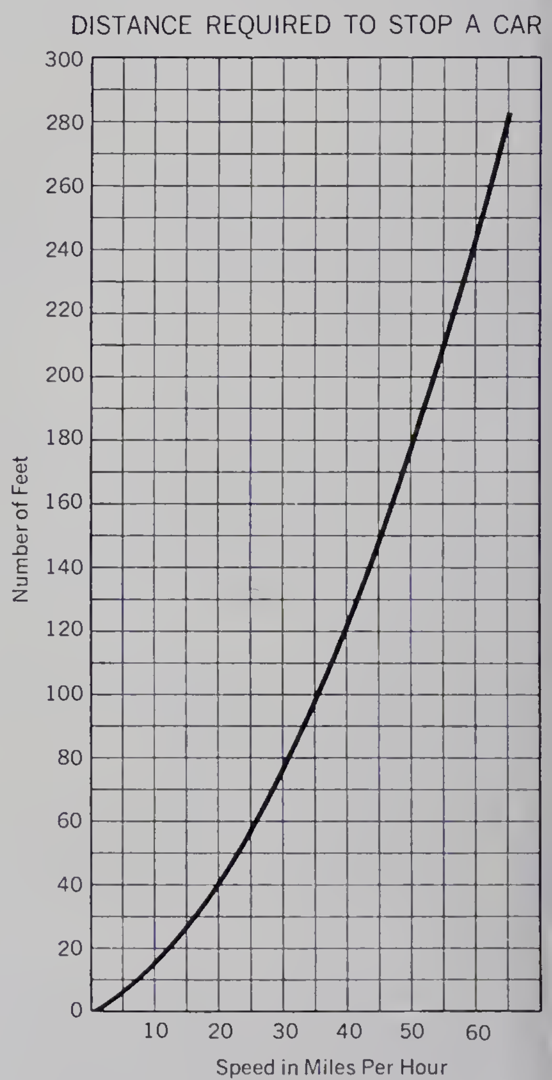
a. 30 m.p.h.? **b.** 40 m.p.h.? **c.** 50 m.p.h.? **d.** 60 m.p.h.?

6. To find the total stopping distance, add the reaction-time distance to braking distance.

m.p.h.	*Equal to ft. per sec.*	*Reaction-time Distance*	*Braking Distance*	*Total Stopping Distance*
10	$14\frac{2}{3}$	11 ft.	5 ft.	16 ft.

Check the above figures: Is the second item, $14\frac{2}{3}$ ft. per sec., correct? Explain how the reaction-time distance was calculated. Complete the table for speeds of 20, 30, 40, 50, and 60 m.p.h.

7. In the Figure below you see a line graph representing stopping distances at different speeds. On a sheet of paper list the numerals 8 through 12. Referring to the graph, select the best answer for each question, and write the letter to indicate your selection.



8. The stopping distance of a car traveling 20 m.p.h. is about:

a. 55 ft. **c.** 30 ft.
b. 40 ft. **d.** cannot tell

9. At 45 m.p.h. the car will stop in about:

a. 200 ft. **c.** 150 ft.
b. 175 ft. **d.** none of these

10. At 60 m.p.h. how far will the vehicle travel before it stops?

a. 325 ft. **c.** 175 ft.
b. 246 ft. **d.** 150 ft.

11. How much farther does a car travel before it stops when traveling 30 m.p.h. than at 20 m.p.h.?

a. 45 ft. **c.** 30 ft.
b. 40 ft. **d.** 36 ft.

12. The difference in stopping distance between a car going at 40 m.p.h. and a car going at 50 m.p.h. is about:

a. 75 ft. **c.** 56 ft.
b. 65 ft. **d.** 45 ft.

A four-sided polygon is called a *quadrilateral.* The prefix *quad-* means four. The suffix *-lateral* refers to the sides. A rectangle is a quadrilateral whose angles are right angles. As in the triangle, the angles are determined by the sides, and the sides of a rectangle are perpendicular to each other.

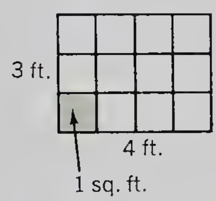
In studying the rectangle, the triangle, and other plane figures, we are concerned with two distinct parts of the figures that should not be confused:

(1) The closed curve defining the figure, as the sides of the rectangle or the triangle, the circle, etc., and
(2) The *region* enclosed by the figure, which is a surface.

The length of the closed curve is its *perimeter* (if it is a polygon) or its circumference (if it is a circle). The measure of the region enclosed by the figure is called its *area.* Both measures are important.

Mr. Brown has a rectangular-shaped cornfield. Its length is 80 rods, and its width is 40 rods. The perimeter is 240 rods. This becomes important if he is planning to fence the field. He says the area is 20 acres. (Later you can verify this.) This measure is important to Mr. Brown when he is calculating how much seed to use, or if he is planning to sell the field.

Study the rectangle below and answer the following questions.



1. The rectangle contains how many square feet in each horizontal row?

2. What is the length of the rectangle?

3. Is the number of feet in the length equal to the number of square feet in one row? (Remember! If the product of the number of feet in the length and the number of feet in the width is equal to one, then we have one square foot.)

4. How many horizontal rows of square feet are there?

5. What is the width of the rectangle?

6. Is the number of feet in the width of the rectangle equal to the number of rows of square feet?

7. Can you find the total number of squares by multiplying the number of squares in one row by the number of rows?

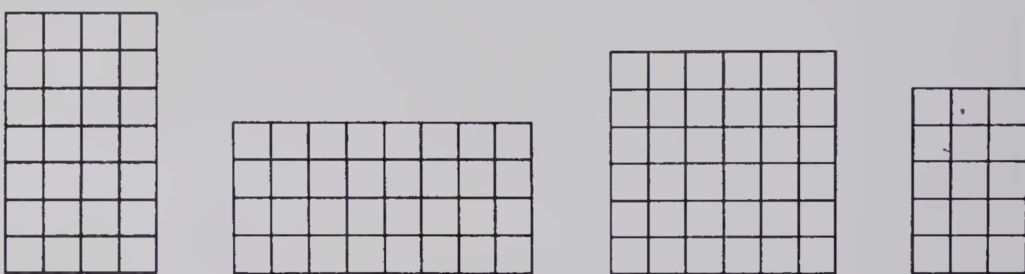
8. What numbers do you multiply?

9. Are these numbers also the number of units in the length and the width of the rectangle?

10. Suppose l represents the number of units (inches, feet, yards, etc.) in the length of a rectangle, and w represents the number of units in the width. How could you express the product? This product tells us the number of square units in the region enclosed by the rectangle.

11. The number of square units in a region enclosed by a geometrical figure is called the *area* of the figure. What is the formula for finding the area of a rectangular region in terms of l and w?

12. Find the area of each of the rectangular regions in the Figures below if each space represents 1 square foot. Use the formula you wrote in Exercise 11: $area = l \times w$



a. b. c. d.

13. Find the areas of the regions enclosed by rectangles having these dimensions.

	Length	*Width*		*Length*	*Width*
a.	6 in.	4 in.	**e.**	4.7 in.	2.3 in.
b.	15 ft.	8 ft.	**f.**	$7\frac{1}{2}$ ft.	$3\frac{2}{3}$ ft.
c.	7 yd.	4 yd.	**g.**	10 rd.	6.4 rd.
d.	20 rd.	15 rd.	**h.**	12 yd.	$5\frac{2}{3}$ yd.

When there is no possibility for confusion, we may refer to the "area of the rectangular region" as the "area of the rectangle." It should not be forgotten, however, that the rectangle consists only of the segments that form the sides.

14. The measurements of Mr. Brown's rectangular-shaped cornfield are 80 rods by 40 rods. What is the area of the cornfield in square rods?

15. An acre is equal to 160 square rods. Is Mr. Brown correct in stating that the area of the cornfield is 20 acres?

If the opposite sides of a quadrilateral have the same measure, then it is a *parallelogram.* (See Figure 1 below.) If the opposite sides of a parallelogram ($\overline{AD}$ and $\overline{BC}$ or $\overline{AB}$ and $\overline{DC}$) are extended indefinitely, they would never intersect. We then say that the opposite sides are *parallel.* Name other examples of parallel sides. We call the side on which the parallelogram is resting the base. The base of $ABCD$ is $\overline{DC}$.

1. Notice the parallelogram in Figure 1 below. If the right triangle AED is removed, we have Figure 2. Now when the triangle is moved over to the right side of the figure as in Figure 3, what is the shape of the figure?

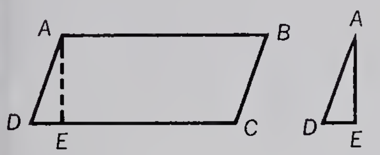


Figure 1

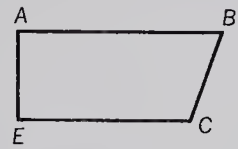


Figure 2

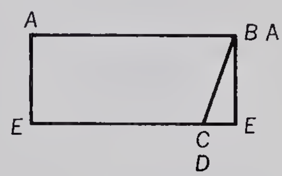
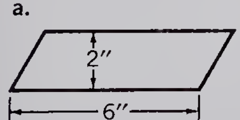
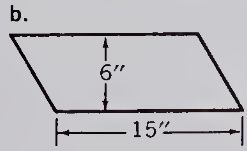
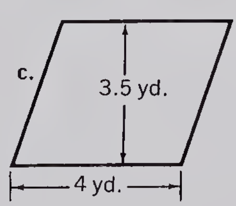


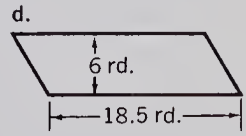
Figure 3

2. The opposite angles of a parallelogram have the same measure. For example, the measures of $\angle A$ and $\angle C$ are the same in Figure 1; $m\angle A = m\angle C$. Name the other pair of opposite angles in Figure 1.
3. Is the area of parallelogram $ABCD$, Figure 1, equal to the area of the rectangle in Figure 3?
4. We know that the area of a rectangle is equal to the product of the length and the width. Is the length of the rectangle, Figure 3, equal to the measure of the base of the parallelogram $ABCD$ in Figure 1?
5. $\overline{AE}$ in Figure 1 is found by drawing a perpendicular from A to $\overline{DC}$. The length of $\overline{AE}$ is the shortest distance between $\overline{AB}$ and $\overline{DC}$ of parallelogram $ABCD$. AE is called the height, h, of the parallelogram. When $\triangle ADE$ is moved as in Figure 3, AE becomes the width of the rectangle. Does it follow that the height, AE, of Figure 1 is the same as the width of Figure 3?
6. Express the formula for the area of the region enclosed by the parallelogram $ABCD$, using b to represent the measure of the base and h to represent the height.
7. Find the area of each of the parallelograms below.

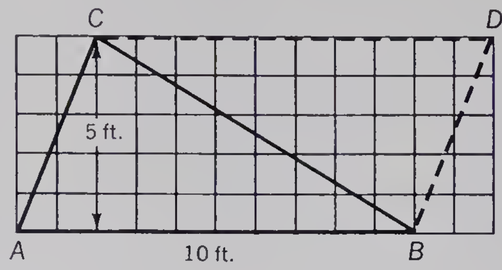
a.


b.


c.


d.


1. In the Figure below, if a triangle congruent to $\triangle ABC$ is cut out and placed in the position of $\triangle BCD$, we have a parallelogram.



a. What are the measures of the base and height of parallelogram $ABDC$?

b. The height, h, of a triangle is the measure of the perpendicular segment from the opposite vertex to the base. What is the measure of the base and the height of $\triangle ABC$?

c. What is the area of the parallelogram?

d. Note that $\overline{CB}$ cuts parallelogram $ABDC$ into two triangles of the same size and shape. $\overline{CB}$ is called the *diagonal* of the parallelogram. The area enclosed by $\triangle ABC$ is what fraction of the area of the parallelogram? What is the area of $\triangle ABC$?

e. How do the areas of $\triangle ABC$ and parallelogram $ABDC$ compare?

f. Therefore, if the formula for the area of a parallelogram is

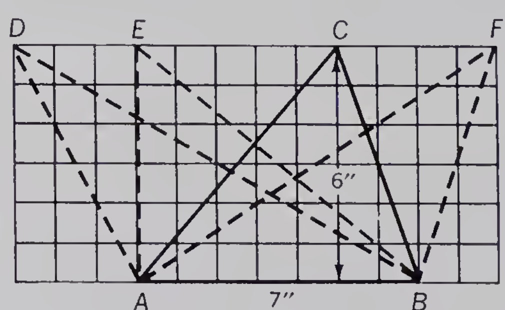
$$A = b \times h \qquad \text{or} \qquad A = bh$$

then the area of a triangle is $A = ?$

2. Find the areas of triangles with these dimensions.

	Base	*Height*		*Base*	*Height*
a.	8 rods	6 rods	**c.**	11 inches	7 inches
b.	19.5 yards	26.7 yards	**d.**	30.8 feet	36.9 feet

3. Regardless of their shape, if two triangles have bases with the same measure and equal heights, do the triangles have the same area? Why? Below you see four triangles having bases with the same measure and equal heights but with different shapes. Find the area of each.



Henry lives on a city block shaped like Figure 1 below. Maple Street is parallel to Elm Street, but 11th and 12th streets are not parallel. The block therefore is shaped like a *trapezoid.* A "trapezoid" is a quadrilateral only two of whose sides are parallel. The parallel sides are the *bases* of the trapezoid.

1. What is the measure of base a in Figure 1? of base b?

2. The shortest distance between the bases is called the height, h. Remember! The shortest distance is the measure of the segment (h) constructed inside the trapezoid that is perpendicular to both bases. What is the measure of h in Figure 1?

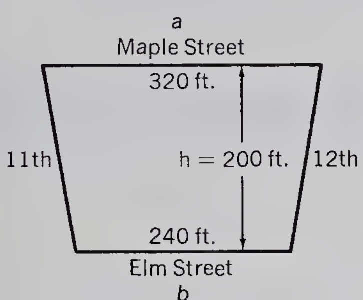


Figure 1

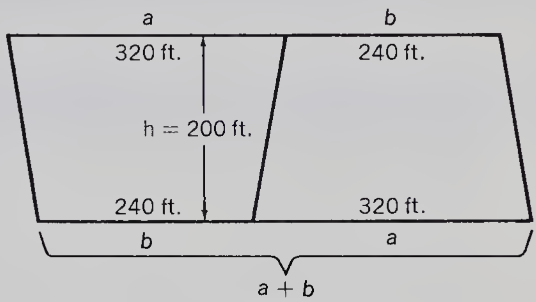


Figure 2

3. Suppose two trapezoids each of the same size and shape as in Figure 1 are placed together as in Figure 2. What geometric figure do they form? Explain why the opposite sides in Figure 2 have the same measure.

4. What are the measures of the base and height of the parallelogram in Figure 2? What is the area of the parallelogram?

5. In general terms, the area of the parallelogram (Figure 2 above) can be stated as

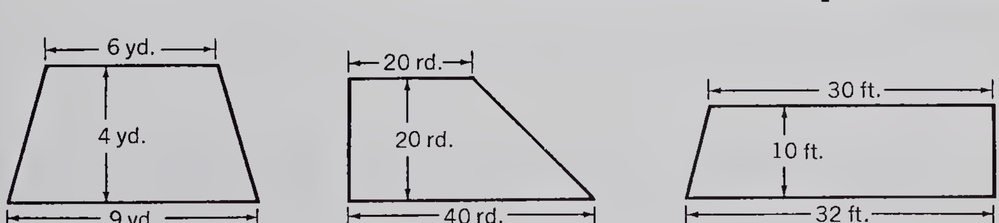
$$A = h \times (a + b) \qquad \text{or} \qquad A = h(a + b)$$

Explain what this means.

6. What is the area of the regions enclosed by each of the trapezoids in Figure 2? The formula for the area of the trapezoid is:

$$A = \tfrac{1}{2} \times h \times (a + b) \qquad \text{or} \qquad A = \tfrac{1}{2}h(a + b)$$

Use this formula to find the area of each of the trapezoids below.



Oral Exercises

Use the following problems to practice the six steps to problem solving. Do as many as you can without using pencil and paper.

STEPS FOR SOLVING APPLIED PROBLEMS

1. Understand the problem.
2. Note what the problem asks for.
3. Look for hidden questions.
4. Estimate a reasonable answer.
5. Set up and solve the conditional statement(s).
6. Check your answer.

1. If a basketball court measures 40 feet by 80 feet, how far would you go in running around the court once? three times?

2. A rectangular-shaped pasture measures 40 rods by 20 rods. How many rods of woven wire fencing are necessary to build a fence around it? The "distance around" the rectangle is called the *perimeter*. In fact, the distance around any polygon is the "perimeter," as we have already mentioned on page 185.

3. What is the perimeter of a rectangular-shaped picture that measures 4 feet by 2 feet?

4. What is the perimeter of a square-shaped patio each side of which measures 10 feet?

5. How many feet must you walk in going around a square-shaped city block that measures 300 feet on each side?

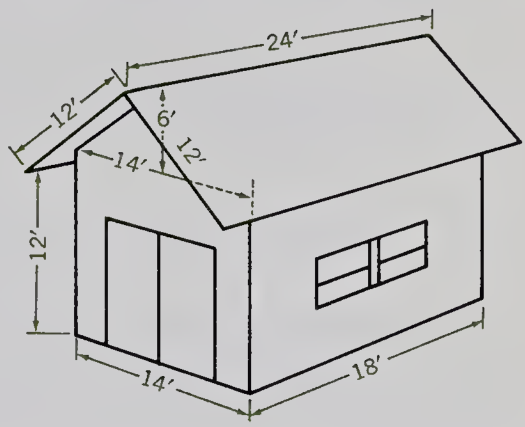
6. How much woven wire would you need to fence a garden shaped like a *regular hexagon*, each side of which is 10 ft. long? A hexagon is a six-sided polygon. (The prefix "hexa-" means six.) The adjective "regular" means that each of the sides has the same measure.

7. How much wire will be needed to fence a garden shaped like a regular *octagon*, each side of which measures 5 feet? An octagon is an eight-sided polygon. What do you think the prefix "octa-" means?

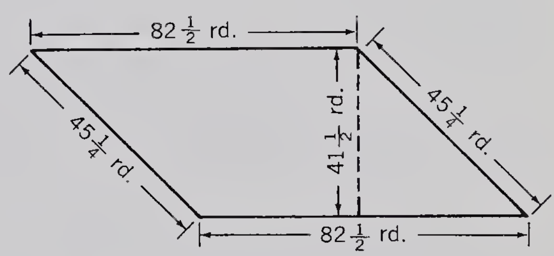
8. At 25 cents per square foot, what will it cost to have a $4' \times 6'$ rug cleaned?

9. A floor that measures 5 yards by 4 yards will be covered by carpeting that costs \$5.00 a square yard. How much will the carpet cost?

1. How many square feet of linoleum are required to cover a rectangular-shaped floor 18 feet long and 12 feet wide?
2. If the linoleum cost \$3 per square yard, what would it cost to cover the floor in Exercise 1?
3. A baseball diamond is a square ninety feet on a side. What is the area in square feet? in square yards?
4. Mr. Olson plans to put a barbed wire fence, with three strands of wire, around his rectangular-shaped pasture. The pasture measures 20 rods by 10 rods. How many rods of barbed wire does he need?
5. If one pound of grass seed is needed for each 500 square feet of lawn, how many pounds are needed to plant a rectangular-shaped lawn that measures 45 feet by 20 feet?
6. How many square feet of roofing material would be required to cover the garage in the Figure at the right?
7. What is the area of each side of the garage? of each end of the garage?
8. A gallon of paint will cover, with two coats, about 300 sq. ft. of surface. How many gallons of paint will be required to put two coats on the sides and ends of the garage, disregarding windows and doors? Give your answer to the nearest gallon.



9. If one roll of wallpaper covers 35 square feet, how many rolls must be bought to cover the four walls of a room 14 feet long, 12 feet wide, and 8 feet high? (Remember that you cannot buy part of a roll.)
10. Mr. Edwards has a field shaped as shown in the Figure at the right. What is the area of the field in square rods?
11. What would it cost Mr. Edwards to put a fence around his field, at \$5.50 per rod?



12. Sod is being transplanted to a rectangular-shaped lot that measures 20 feet by 50 feet. The price of sod is 10 cents per square foot. How much will the sod cost?
13. A square-shaped field measures 20 rods on a side. What is the area of the field?

If we cut a circular region into two pieces of the same size, we have two *semicircular regions.* If we then cut each of these regions into small sectors or pieces, like pieces of a pie, it would look like Figure 1a below.

If you think of these cuts continuing through the circular region but not through the circle, you could "stretch" the semicircles out as in Figures 1b and 1c.

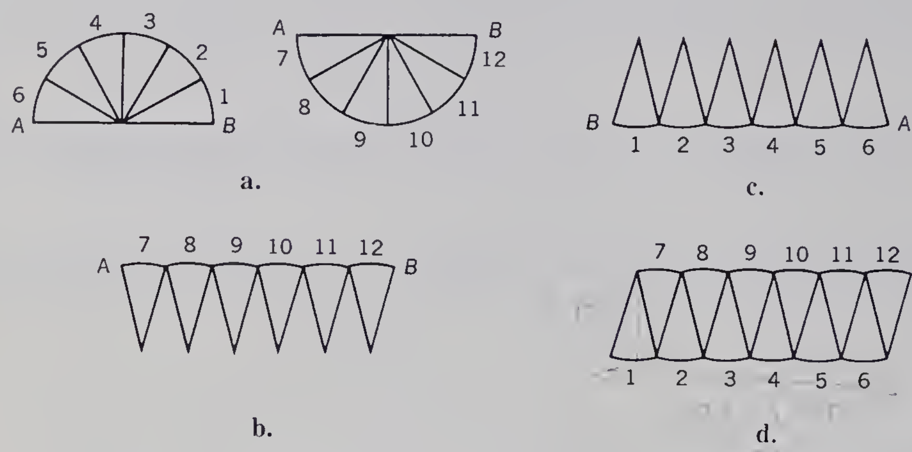


Figure 1

Considering the two semicircular regions, you now have two pieces. Now you could fit Figure 1b into Figure 1c. See Figure 1d. The resulting figure is very much like a parallelogram. Can you see that the height of this "parallelogram" is equal to the measure of the radius of the circle? The measure of the base of the parallelogram is equal to $\frac{1}{2}$ the circumference of the circle. Why?

To find the area of a parallelogram we multiply the measure of the base by the height. In the case of Figure 1d, the area formula would be $A = r \times \frac{1}{2}c$, where r represents the height and $\frac{1}{2}c$ the base. We have already learned that the formula for the circumference is $c = \pi \times d$. Thus the formula for one-half the circumference would be:

$$\tfrac{1}{2}c = \tfrac{1}{2} \times \pi \times d$$

We can now replace $\frac{1}{2}c$ in the area formula $A = r \times \frac{1}{2}c$:

$$A = r \times \overbrace{\tfrac{1}{2} \times \pi \times d}^{\frac{1}{2}c}$$

We also learned that the diameter, d, is related to the radius, r, by the formula $d = 2 \times r$. This means we can replace d in the area formula $A = r \times \frac{1}{2} \times \pi \times d$:

$$A = r \times \tfrac{1}{2} \times \pi \times \overbrace{2 \times r}^{d}$$

By using the commutative and associative properties this becomes:

$$A = r \times \overbrace{\tfrac{1}{2} \times 2}^{1} \times r \times \pi$$

Since $\frac{1}{2} \times 2 = 1$:

$$A = r \times r \times \pi$$ or by the associative and
$$A = \pi \times r \times r$$ commutative properties

Since $r \times r = r^2$:

$$A = \pi \times r^2$$

or

$$A = \pi r^2$$

This is the formula for the area of the region enclosed by the circle. We will call this the area of the circle.

1. What is the area of a circle whose radius measures 21 inches?
2. A radius of the free throw circle on a basketball court measures 6 feet. What is the area of the circle?
3. What is the area of a circular mirror whose radius measures 15 inches?
4. A radius of the jumping circle on a basketball court measures 2 feet. Find its area.
5. What is the area of a circular television tube whose diameter measures 17 inches?
6. How much greater is the measure of a radius of the "outer" circle in Figure 1 below than one of the "inner" circle?
7. Find the circumference of each circle.
8. Find the area of the outer circle.
9. Suppose that Figure 1 below represents a walk around a circular pool that measures 12′ in diameter. What will it cost to surface the walk at $2.50 per square yard of surface area?
10. A circle is to be cut from a square the measure of whose sides is each 4 inches. How much of the area of the square will be wasted (Figure 2 below)?
11. At 25 cents per square foot, what will it cost to cover the semicircular portion of Figure 3? the rectangular portion?

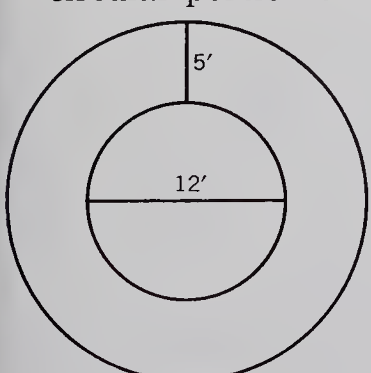


Figure 1

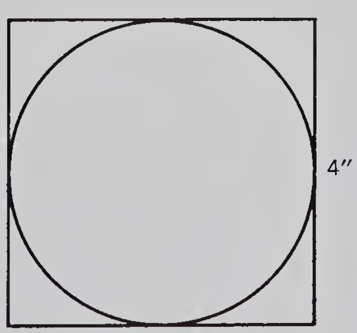


Figure 2

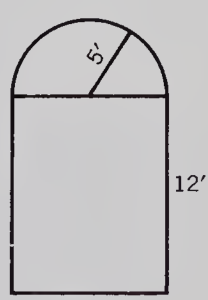


Figure 3

Mr. Manson had a large farm in Iowa. The state planned a new highway which would cut into his south quarter section. The new road and the existing fences shown in dotted lines (Figure 1 below) divided the land into 4 plots. Mr. Manson needed to know the size of each plot.

1. Mr. Manson hired a surveying student to help him measure the land. Their measurements in rods are given on the drawing in Figure 1. What is the area of Plot B in acres? (160 square rods = 1 acre.)
2. How many acres are in Plot C?
3. Mr. Manson divided Plot A into two parts by extending the 80-rod line to the right. How many acres are there in each of the parts? How many acres are in Plot A?

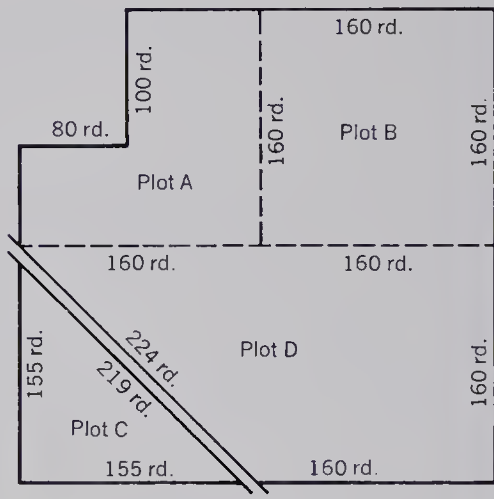


Figure 1

4. The surveying student was able to calculate the area of Plot D from the measurements. Can you? What is the area of Plot D?
5. How much of Mr. Manson's property was taken up by the road?
6. Study carefully the several geometric figures in Figure 2. See if you can observe similarities and differences in groups of these figures. Two common properties of Figures b, c, and d are that they have the same number of sides and angles. How many angles does each of them have?

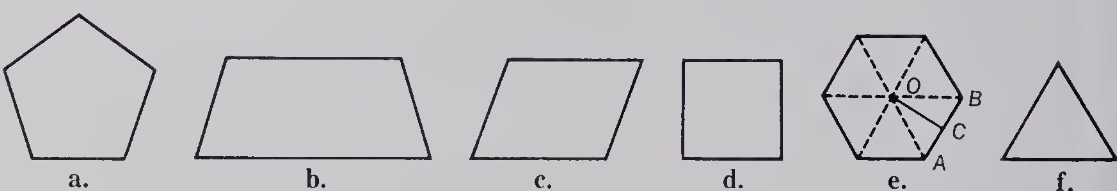


Figure 2

7. A common property of Figures a, d, e, and f above is that in each one, all of the sides have the same measure. Figure a is a *regular pentagon.* A pentagon is a five-sided polygon. What does the adjective regular mean? What does the prefix "penta-" mean? Name Figures d, e, and f.
8. Which Figures have their opposite sides parallel?
9. We can find the area of Figure e if we know how to find the area of which part of the Figure?

10. In Figure e, if AB is 12 inches long and OC is 10.4 inches, what is the area of the hexagon?

11. Suppose that the bases of Figures c and f each measure 24 units, and the heights are each 20 units. Compare the cost of covering each Figure with cloth that costs 22 cents per square unit.

12. The floor space of a bathroom is to be covered with square tiles. The space to be covered measures 9 feet by 8 feet. How many tiles measuring 3 inches on a side will be needed to cover the floor space?

13. Is your answer to Exercise 12 exact or approximate? Explain.

14. A given regular hexagon has sides each measuring 11.6 feet (Figure 3). The perpendicular distance from the center to each side measures 10 feet. How many square feet are in the hexagon?

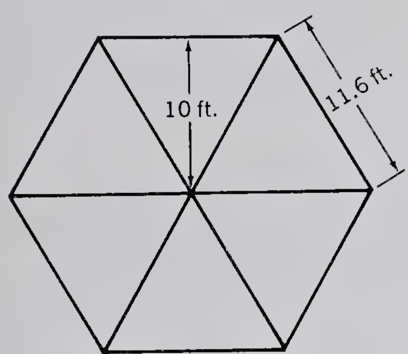


Figure 3

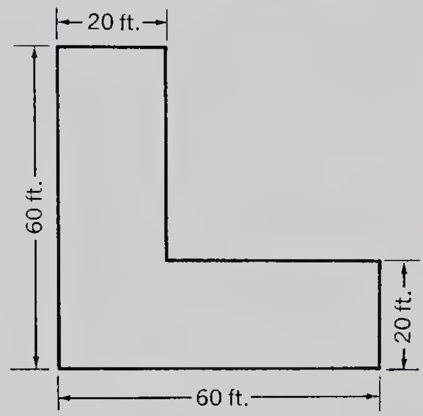
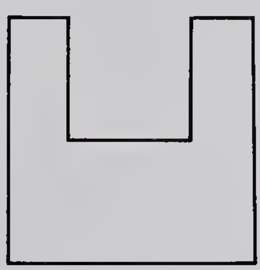


Figure 4

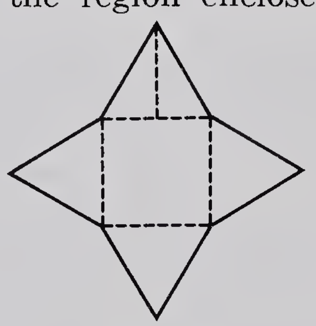
15. An apartment building was built in the "L" shape shown in Figure 4 above. What is the area of the ground covered by the building?

16. The Figures below are drawn to *scale*, as shown. Measure each Figure to find its dimensions, and then calculate the total area (to the nearest hundredth, if necessary) of the region enclosed by each Figure.



1 in. = 100 yds.

a.



1 in. = 1 ft.

b.

Remember! In constructing graphs, you had to determine the "key." By saying that 1 inch = 100 yards, we actually mean that 1 inch on the drawing represents 100 yards. This is the "key" of the scale drawing. The scale drawing represents the actual picture "shrunken" down to a workable size.

Planes, Lines, and Points

A *plane* is a set of points that makes up a flat surface extending without limit in all directions. For example, we can think of a flat surface like a sheet of paper or a table top as representing a portion of a plane. A plane can also be thought of as a set of lines.

1. In Figure 1, you see two lines on the plane A. We will use a sheet of paper to represent the plane and pencil marks to represent lines.

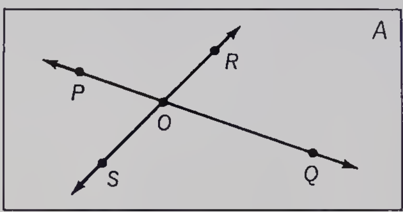


Figure 1

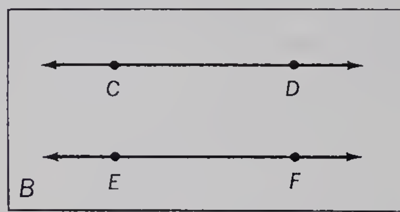


Figure 2

On a sheet of paper draw two intersecting lines, and label the point of intersection O, as in Figure 1. Complete this statement:

$$\overleftrightarrow{PQ} \cap \overleftrightarrow{RS} = ?$$

2. The lines, $\overleftrightarrow{EF}$ and $\overleftrightarrow{CD}$, in Figure 2 are parallel lines, since they will not intersect if extended indefinitely. Therefore, do the lines have any points in common?

3. Remember! When the number of elements in a set is zero, the set is said to be empty. An "empty set" is represented by the symbol ϕ or $\{\ \}$. Complete this statement for Figure 2:

$$\overleftrightarrow{CD} \cap \overleftrightarrow{EF} = ?$$

4. In Figure 3, line m is not contained in plane C. It intersects the plane C at point O. Complete this statement:

$$\text{plane } C \cap \text{line } m = ?$$

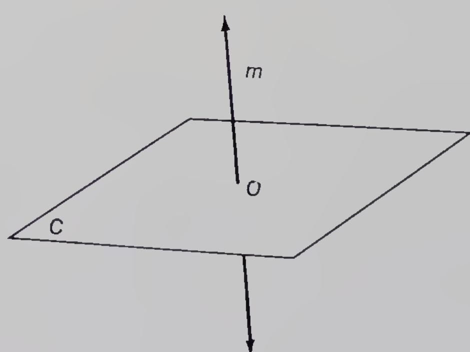


Figure 3

5. In Figure 4, you see the two intersecting planes, A and B. What geometric figure is formed by their intersection?

6. Complete this statement:

$$\text{plane } A \cap \text{plane } B = ?$$

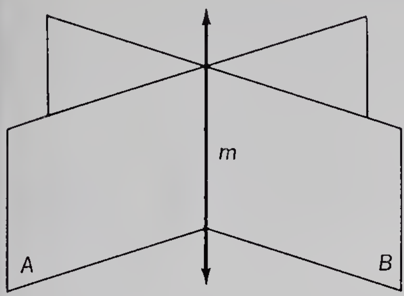


Figure 4

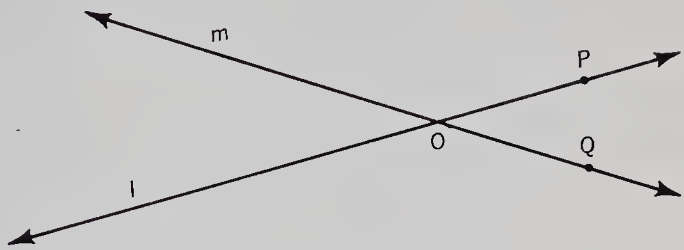


Figure 5

7. If planes A and B did not intersect then they would be parallel. Complete the following statement if planes A and B are parallel:

$$\text{plane } A \cap \text{plane } B = ?$$

8. Draw two intersecting lines, l and m, as in Figure 5. Draw a third line through a point P on line l and a point Q on line m. If you continued passing lines through pairs of points on each of the original lines, what kind of geometric figure would you determine?

9. From Exercise 8, what can we conclude about two intersecting lines?

10. Could you have two non-parallel lines that do not determine a plane? Explain your answer, using two pencils to illustrate.

11. Represent three points on your paper, as A, B, and C in Figure 6, that are not on the same line. Draw a line through points A and B and one through points A and C. Have you determined a plane? Explain your answer.

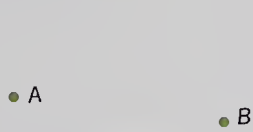


Figure 6

12. Explain why three points that are not collinear determine a plane.

13. Refer to Figure 6 and explain this statement: A plane is determined by a line and a point not on the line.

14. A 4-legged stool frequently will wobble, whereas a three-legged stool will not wobble. Use Exercise 12 to explain why.

15. Sketch an example in the classroom of a representation of:
    - **a.** Two planes that have a line in common
    - **b.** Three planes that have a point in common
    - **c.** Three planes which have a line in common

A. Insert the decimal point in the proper place in the product. Annex zeros when necessary.

1. $17.5 \times .05 = 875$
2. $59.5 \times 7.2 = 42840$
3. $1.5 \times 2.5 = 375$
4. $.017 \times .05 = 85$
5. $2.9 \times .034 = 986$
6. $6.38 \times .015 = 9570$
7. $8.63 \times .19 = 16397$
8. $3.3 \times 0.45 = 1485$
9. $1.09 \times 21.6 = 23544$
10. $45.3 \times 1.08 = 48924$

B. Find the products.

1. $19 \times .7$
2. $1.6 \times .6$
3. $.25 \times .3$
4. $17 \times .05$
5. $14 \times .7$
6. $.55 \times .3$
7. $2.7 \times .4$
8. $1.5 \times .06$
9. 5.8×9
10. $7.38 \times .005$
11. 7.25×56
12. 5.09×6.13
13. $.27 \times .38$
14. 8.05×9.075

C. Insert the decimal point in each quotient.

1. $15.6 \div .04 = 39$
2. $4.64 \div 1.6 = 29$
3. $.567 \div .009 = 63$
4. $57.6 \div .012 = 48$
5. $0.936 \div 1.8 = 52$
6. $0.648 \div .09 = 72$
7. $1.728 \div 6 = 288$
8. $.1925 \div .005 = 385$
9. $91.26 \div .09 = 1014$
10. $44.4 \div .037 = 12$
11. $98.6 \div .034 = 29$
12. $6.543 \div 0.18 = 3635$

D. Find the quotients.

1. $2.1 \div 7$
2. $4.5 \div .5$
3. $.96 \div 6$
4. $18 \div .09$
5. $1.96 \div 1.4$
6. $22.5 \div .045$
7. $72 \div .18$
8. $.256 \div .32$
9. $1.728 \div 1.44$
10. $1.995 \div .0105$
11. $4.005 \div .45$
12. $256.608 \div 6.48$
13. $37.73 \div 7.7$
14. $11.0088 \div 8.34$

If you need more practice, turn to page 463 and following. If not, you may work in the Experts' Corner which follows.

Another Kind of Magic Square

In an earlier chapter you learned how to make a magic square with an odd number of cells along the side. You can also make one with four cells on a side by using a different method.

1	5	9	13
2	6	10	14
3	7	11	15
4	8	12	16

Figure a

16			4
	11	7	
	10	6	
13			1

Figure c

	5	9	
2			14
3			15
	8	12	

Figure b

16	5	9	4
2	11	7	14
3	10	6	15
13	8	12	1

Figure d

Figure d is a completed magic square with 4 cells on a side. Figures a, b, and c show how it was made. The Steps on the following page will explain what was done in Figures a, b, c, and d.

1. Starting with 1 in the upper left cell, write numerals down the columns, ending with 16 in the lower right cell.

2. Draw the two diagonals (Figure a).

3. Fill in the cells in Figure b that are not crossed by the diagonals.

4. Reverse the numerals in each diagonal. Write the numeral in the lower right in the upper left cell, and the other numerals in order so the 1 is in the lower right cell. Reverse the numerals in the other diagonal in the same way (Figure c).

5. In Figures b and c separate squares were used for these operations, so you could see them more clearly. You can use one square for both operations, and get your magic square at once (Figure d).

1. Find the sum of the numbers named in each row, each column, and each diagonal in Figure d. Is it a magic square?

2. Practice until you can make the magic square without referring to the directions or the Figures.

3. The formula for the sum of each row and diagonal, when you use consecutive numbers beginning with 1, is, for any magic square:

$$S = \frac{n}{2}(n^2 + 1)$$

where S is the sum, and n is the number of cells on a side. Does the formula hold for $n = 4$?

4. Without referring to the Figures or Steps, construct a magic square of 4 cells on a side using the even numbers: 2, 4, 6, etc.

5. Construct a magic square, 4 cells on a side, using an arithmetic progression of thirds: $\frac{1}{3}$, $\frac{2}{3}$, $\frac{3}{3}$, etc.

6. In the first chapter there was a puzzle in which you were asked to work nine problems whose answers gave you a magic square. Prepare a similar puzzle for a magic square with 4 cells on a side. You may use any arithmetic progression you wish for the numbers.

SPECIAL PROJECTS

You will find a great deal of interesting information about magic squares in encyclopedias and in books on mathematical recreations and puzzles. Prepare a report on one or more of the following topics.

1. What people were interested in magic squares in ancient times?

2. What magic powers did they attribute to magic squares?

3. What were some important and interesting magic squares that they developed?

4. What other methods of forming magic squares can you find besides the ones given here?

Thus far we have been concerned with finding the area of regions enclosed by various geometric figures. Area involves two dimensions, length and width (or height). In the following problems we will be concerned with *volume* and will be discussing a *third* dimension — depth. We have used height to mean width, but for this discussion we will use height to mean depth.

1. Suppose that you have a box of candy in which each piece is shaped like a *cube* that measures one inch on each side. A cube is a three-dimensional geometric figure that has six *faces*, like a box. To be a cube, each face must have the same dimensions. Each face then is a square. The box is one inch deep, and the bottom measures one foot, on each side. Therefore, the box contains one layer of candy. How many pieces could be packed in the box? (See Figures 1a and 1b below.)

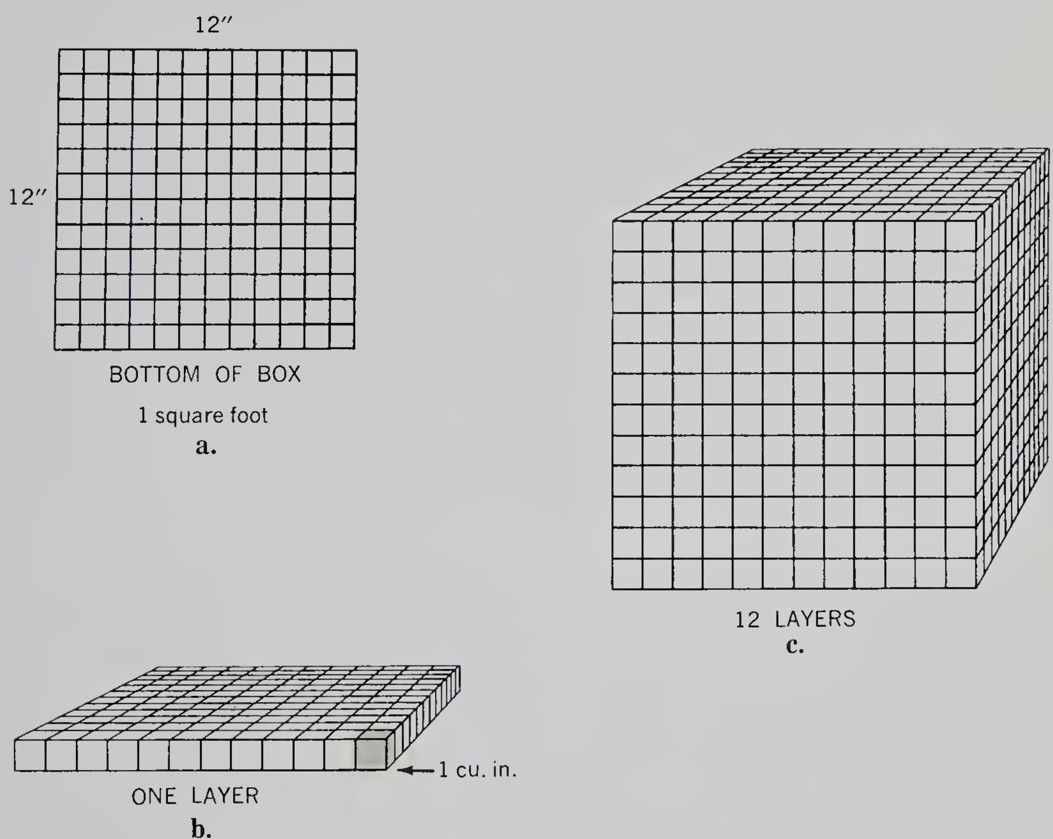


Figure 1

2. Suppose we have a box large enough so that we can pack twelve layers of candy. How many inches would the layers measure in height? (See Figure 1c.)

3. Does the box with the twelve layers represent a cube? Explain your answer.

4. How many pieces of candy would be in the box? Since each piece of candy measures one inch on each side, we say that each piece of candy represents one *cubic inch.* Since the box measures one foot on each side, we say the box represents one *cubic foot.* Therefore, the number of pieces of candy in the box represents the number of cubic inches in a cubic foot. How many cubic inches are there in a cubic foot? When we measure in cubic units, we are measuring *volume.*

5. Use Figure 2 to explain how many cubic feet are in a cubic yard.

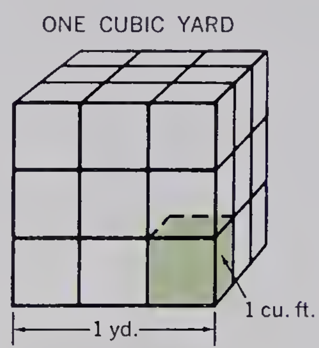


Figure 2

6. A *rectangular solid* is a three-dimensional geometric figure that has six faces. Each face is shaped like a rectangle, and faces opposite each other have the same dimensions. You can use the general method of the previous exercises to find the volume of the interior of any rectangular solid. A grain bin measures 18′ long, 9′ wide, and 6′ high. How many cubic feet does it contain (Figure 3)?

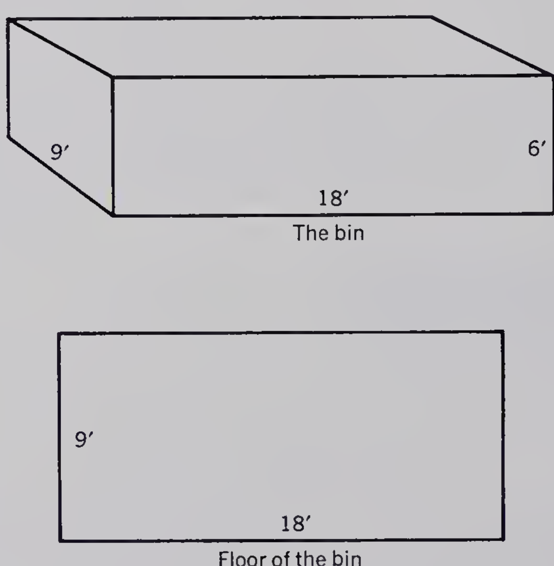


Figure 3

a. What is the area of the floor of the bin?
b. How many cubic feet would the bin contain if it were 1 foot high?

c. The bin is 6′ high. How many 1-foot layers would there be?
d. How many cubic feet would this be?

7. State the rule expressed by this formula for finding the volume of the interior of a rectangular solid:

$$V = l \times w \times h = lwh$$

8. Use the formula to find the number of cubic feet in a box that measures 8′ by 4′ by 6′.

Find the volume of boxes with the following dimensions.

9. $8'' \times 6'' \times 3''$

10. $3.5' \times 6.0' \times 4.3'$

11. $2.1'' \times 5.5'' \times 8.0''$

12. $4\frac{1}{2}'' \times 6'' \times 10\frac{1}{4}''$

13. $2\frac{1}{4}' \times 6' \times 9\frac{1}{2}'$

14. $4.5' \times 2.4' \times 2.0'$

15. $6.4' \times 4.0' \times 5.1'$

16. $3\frac{3}{4}'' \times 8\frac{1}{2}'' \times 5\frac{1}{4}''$

17. $6' \times 8' \times 1'$

18. $3' \times 4\frac{1}{2}' \times 7\frac{3}{4}'$

19. Which will hold more, a box 4′ by 3′ by 2′ or a box which is a 3-foot cube?

20. How many cubic yards of earth were removed from a cellar that measures 20′ by 18′ by 9′?

HINT: Change the feet to yards and then multiply, or you can multiply the given dimensions and divide the product by 27. What does 27 represent? Use both methods and decide which is easier. Round the answer to the nearest whole number.

21. The basement under Mr. Jones' new home is to be 30′ long, 24′ wide, and 9′ deep. The earth will be hauled off in truckloads of 3 cubic yards each. How many truckloads of earth will need to be hauled off?

22. A cubic foot of water is equivalent to about 7.5 gal. How many gallons are there in a rectangular-shaped tank that measures 6′ by 3′ by 2′?

23. A cubic foot of water weighs about 62.5 lb. What is the weight of the water in the tank in Exercise 22?

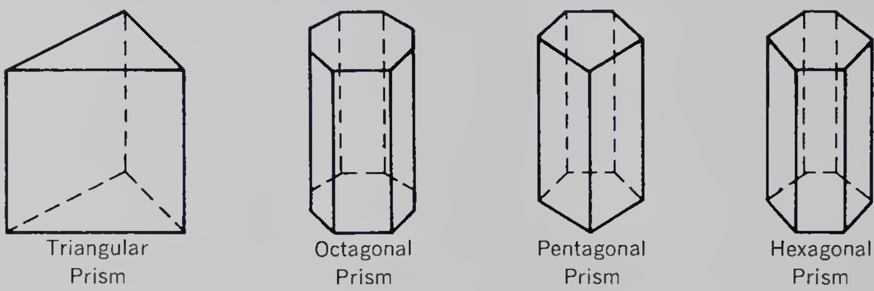
24. How many cubic feet of space are there in a schoolroom that measures 36.0′ by 24.0′ by 12.5′? Allowing 300 cu. ft. of space for each pupil, this room is large enough for how many pupils?

25. Get the measurements of your classroom to the nearest foot. How many cubic feet of space are there for each pupil?

26. Mr. Allen wants to build a storage bin that will hold 180 cubic feet of corn. If the bin is to have a square base, 6 feet by 6 feet, how high must it be?

Faces and Vertices of a Prism

The Figures below illustrate four types of geometric figures called *prisms*. The type of prism is named according to the shape of its *bases*. The two bases of a prism always have the same shape and size, and are parallel to each other. The other *faces*, called *lateral* (meaning side) faces, are always parallelograms.



1. Are the two bases in each of the prisms identical in shape and size?
2. How many lateral faces does a *hexagonal* prism have?
3. How many faces does a *triangular* prism have, including bases?
4. Can you tell how many lateral faces a *pentagonal* prism has, without counting?
5. The lines where the faces of a prism meet are called the edges of the prism. What is the total number of edges of a triangular prism?
6. How many edges (total) does a hexagonal prism have?
7. How many edges (total) does a *rectangular* prism have?
8. Can you tell how many edges an *octagonal* prism has without counting? How many?
9. There is no example, among the Figures, of a "trapezoidal" prism. Can you draw one? Remember! The bases are trapezoids, and the lateral faces are parallelograms.
10. What examples of prisms can you find in the school room? Make a list of the prisms you find, and state what kind they are.
11. All of the prisms shown above are *right* prisms, because their faces, other than the bases, are rectangles. Do you find any prisms in the classroom that are not right prisms?
12. The corners of a prism are called *vertices*. The singular is *vertex*. Do all rectangular prisms have the same number of vertices? How many do they have?

You found the volume of a rectangular prism by using the formula $V = lwh$. Since the product $l \times w$ represents the area of the base (Figure 1a below), the rule for finding the volume could be stated:

Volume = area of the base times the height, or
$V = Bh$, where B is the area of the base.

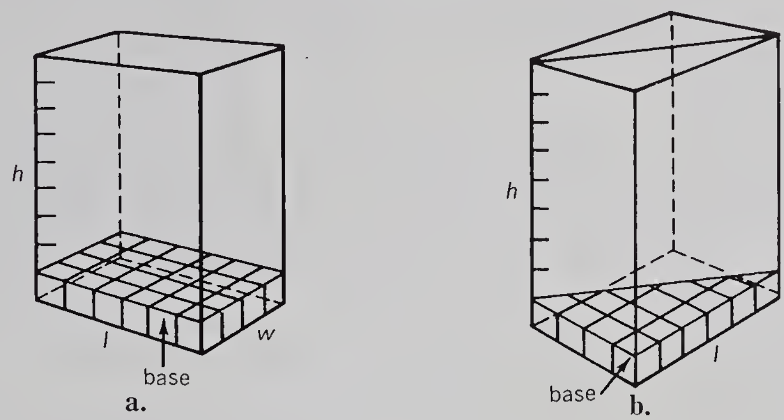


Figure 1

If you study Figure 1b, you can see that this formula can also be used to find the volume of a triangular prism. When we speak of the volume of a solid, we mean the volume of the interior. This is similar to our previous work with area, where we spoke of the area of the geometric figure and meant the area of the region enclosed by the figure. The area of the base tells you how many cubic units can be fitted on the bottom layer. As Figures 1a and 1b illustrate, the area of the base of a triangular prism is equal to one-half the area of a rectangular prism having the same length and width as the triangular prism. The height tells you the number of layers there will be.

1. In Figure 1a, $l = 6$, $w = 4$, and $h = 9$. What is the volume of the rectangular prism?

2. In Figure 1b, $l = 6$, $w = 4$, and $h = 9$. What is the volume of the triangular prism?

3. In Figure 2 you can see that the base of the hexagonal prism is made up of six triangles. The area of each triangle is $A = \frac{1}{2}ab$. If $a = 7$ and $b = 8$, what is the area of the base of the prism?

4. If $h = 12$, what is the volume of the prism?

5. A cubical packing box is 20″ high. How many packages each measuring $4'' \times 4'' \times 10''$ can be packed in one box?

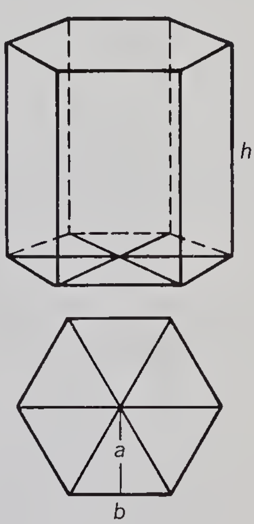


Figure 2

Frequently we need to measure the *capacity* of a bin in bushels, to determine how much grain it will hold. Or we may need to find the capacity of a tank in gallons. Units of dry and liquid measure are used for this purpose.

Dry Measure	*Liquid Measure*
1 bushel (bu.) = 4 pecks (pk.)	1 gallon (gal.) = 4 quarts (qt.)
1 pk. = 8 quarts (qt.)	1 qt. = 2 pints (pt.)
1 qt. = 2 pints (pt.)	1 pt. = 16 fluid ounces (oz.)

1. A bushel is legally defined in the United States as containing 2150.42 cu. in. How many cu. in. are there in 1 qt. dry measure?

2. A gallon contains 231 cu. in. How many cu. in. are there in 1 qt. liquid measure?

3. If a merchant measured 8 gallons of grain as a bushel, how many cu. in. too much or too little would there be?

4. When you find it convenient you may use the approximation 7.5 gal. = 1 cu. ft. How many cu. in. more or less than 1 cu. ft. are there in 7.5 gal.?

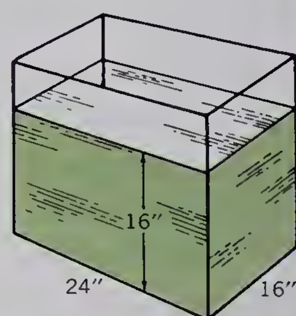
5. One cu. ft. of water weighs 62.5 lb. What does 1 gal. weigh, to the nearest tenth of 1 lb.?

6. What does 1 pt. of water weigh, to the nearest hundredth of 1 lb.?

7. A fluid ounce is a measure of capacity, not of weight. To the nearest hundredth of 1 oz., what does a fluid ounce of water weigh?

8. Another useful approximation is that 1 bu. = 1.25 cu. ft. How many cu. in. is 1 bu. more or less than 1.25 cu. ft.?

9. The aquarium in the Figure at the right is filled to a depth of 16″. How many gallons of water are in the aquarium?



10. To the nearest tenth of a pound, what is the weight of the water in the aquarium?

11. A rectangular bin 12′ long and 6′ wide is filled with wheat to a depth of 5′. How many bushels of wheat does the bin contain?

12. A bin measures 15′ long and 10′ wide. To what depth must it be filled to hold 600 bu. of wheat?

13. A freight car measures 40′ long and $8\frac{1}{2}$′ wide. It is to be loaded with 2000 bu. of wheat. To what depth should it be loaded?

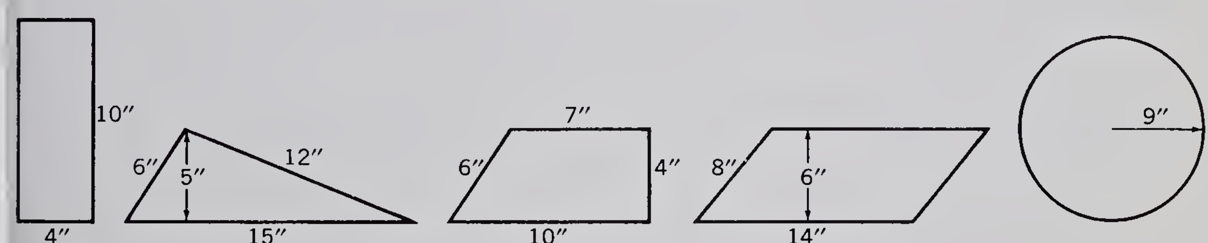
14. Mr. Adams has a corncrib that is 20′ long and 8′ wide. It is filled to a depth of 9′. How many bushels of corn does it hold?

Part One

A. Each of the descriptions below defines one of the words on the right. On your paper write the numerals 1 through 10. Find the word that goes with each description and write the letter corresponding to that word next to that numeral on your paper.

1. A four-sided polygon whose opposite sides are parallel	**a.** arc
	b. chord
2. The segment cutting a parallelogram into two congruent triangles	**c.** circle
	d. diagonal
3. A part of the circumference of a circle	**e.** diameter
4. Multiplying a number by itself	**f.** hexagon
5. A closed curved figure every point of which is the same distance from the center	**g.** hypotenuse
	h. parallelogram
6. A unit of area	**i.** right triangle
7. The longest chord in a circle	**j.** semicircle
8. A triangle with one right angle	**k.** square foot
9. A segment joining any two points on a circle	**l.** squaring
10. A four-sided polygon	**m.** quadrilateral

B. Write the letters a to e on a sheet of paper. After each letter, for each of the Figures below: (1) state the name of the figure, (2) give the formula for its area, and (3) compute its area.



a. b. c. d. e.

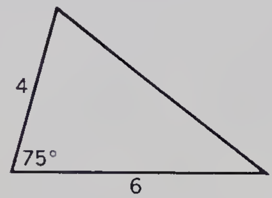
C. The two columns on the right below give a list of formulas. On the left is a list of areas. Write the numerals 1 through 6. After each copy the formula, or formulas, that will give you the area desired.

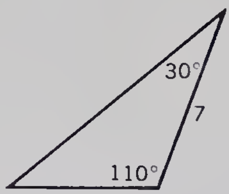
1. Area of a circle	$A = \pi r^2$	$A = lw$
2. Area of a parallelogram	$A = bh$	$A = 4s$
3. Area of a rectangle	$A = 2\pi r$	$A = \pi dl$
4. Area of a square	$A = h\dfrac{(a+b)}{2}$	$A = \frac{1}{2}ab$
5. Area of a trapezoid		$A = s^2$
6. Area of a triangle		

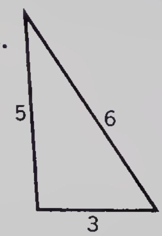
D. In the Figures below, each triangle in Column I is congruent to one of the triangles in Column II. List the numerals 1 to 5 on a sheet of paper. After each numeral write the letter to show which triangle in Column II is congruent to the triangle having that numeral in Column I. Then, to show which of the three congruence facts makes the triangles congruent, write:

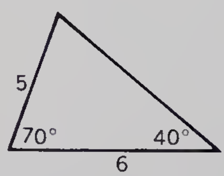
SAS, if two sides and the included angle have the same measure.
ASA, if two angles and the included side have the same measure.
SSS, if the three sides of one are equal in measure to the three sides of the other.

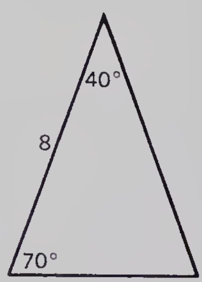
Column I

1.


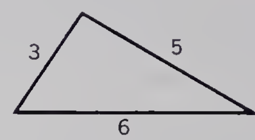
2.


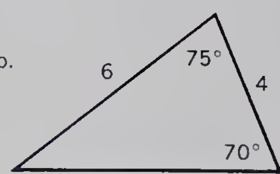
3.


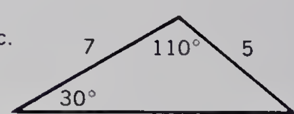
4.


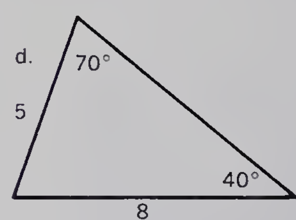
5.


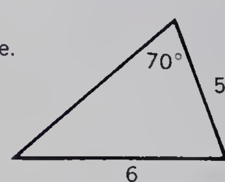
Column II

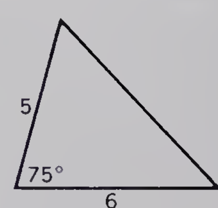
a.


b.


c.


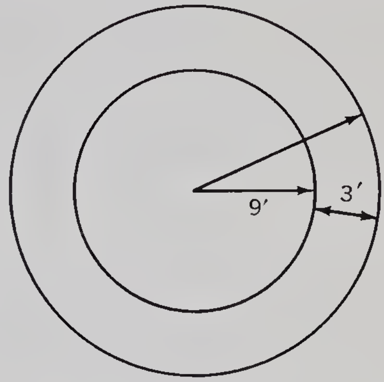
d.


e.


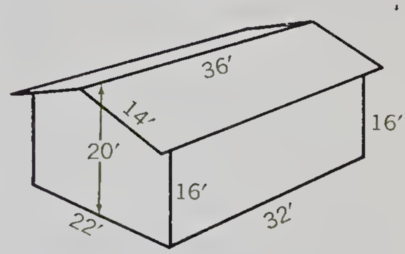
f.


Part Two

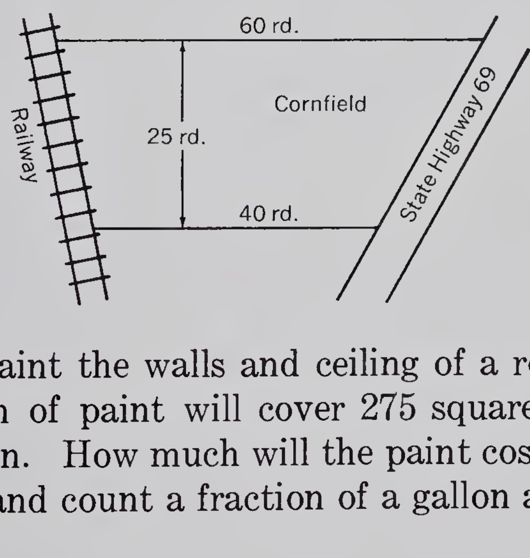
1. Mr. Jones has a rectangular water tank whose base measures $3' \times 6'$. It is $5'$ high. How many gallons will it hold when it is half full?
2. A radius of a circular flower bed is $5'$. What is the area of the flower bed? What is the circumference of the flower bed?
3. A sidewalk $3'$ wide is to be laid around a circular-shaped garden that measures $18'$ in diameter. How many square feet of surface will be required for the sidewalk? (The area of the sidewalk is the difference between the areas of the outer and the inner circles in the Figure at the right.)



4. A badminton court of regulation size measures $20' \times 44'$. The net crosses the court at the center line, which is $22'$ from each end line. Make a scale drawing of the court, showing the center line.
5. Mr. Mason is planning to paint his barn, shown in the Figure at the right. How many square feet of surface are there on each end of the barn?



6. How many square feet are there on the two sides (total)?
7. A gallon of paint will cover 250 square feet. How much will the paint cost at \$4 a gallon? (Count a fraction of a gallon as a gallon.)
8. Mr. Jenkins' cornfield is shaped like a trapezoid, as shown in the Figure below. How many acres are in his cornfield? (160 square rods = 1 acre.)



9. Jim plans to paint the walls and ceiling of a room, $9' \times 13'$ and $8'$ high. A gallon of paint will cover 275 square feet of surface and costs \$5 a gallon. How much will the paint cost? (Disregard doors and windows, and count a fraction of a gallon as a gallon.)

RATIO AND PROPORTION

WORDS TO WATCH FOR

denominator	*per cent*	*ratio*	*sequence*
extremes	*progression*	*scale*	*similar figures*
means	*proportion*	*scale drawing*	*terms*
numerator	*rate*		

Numbers may be compared either by subtraction or division, depending on the kind of comparison you wish to make. For example, Jim and Harry worked last summer in a factory. Jim earned \$320, and Harry earned \$240. Let us make some comparisons.

1. Harry earned how much less than Jim? (Does this call for division or subtraction?)

2. Jim's earnings were how many times as great as Harry's? (Which operation does this call for?)

3. The number of dollars Harry earned was what fraction of the number of dollars earned by Jim?
Note: This question calls for division, with the quotient expressed as a fraction.

4. The fraction expressing the quotient in Exercise 3 names the *ratio* between Harry's earnings and Jim's earnings. Which number is the denominator in Exercise 3 before writing in simplest form?

5. Express the ratio of Jim's earnings to Harry's earnings. Which number is the denominator?

6. Bill weighs 135 pounds and Eric weighs 150 pounds. What is the ratio of Bill's weight to Eric's weight?

7. What is the denominator in the comparison, in Exercise 6, before writing in simplest form?
8. Express the ratio of 64 to 96 in simplest form. Which number is the denominator before writing in simplest form?
9. A field measures 120 rods long and 80 rods wide. What is the ratio of length to width?
10. What is the ratio of the width of the field to its length?

In each of the following problems, state first which number is the denominator before writing in simplest form. Then express the ratio in simplest form.

11. Jim had a 16-foot board. He used 10 feet for a shelf. What part of the board was left?
12. A classroom measures 20 feet wide and 26 feet long. What is the ratio of the length to the width?
13. One year 24 girls and 21 boys received scholarship awards at Adams High School. What fraction of those receiving awards were boys?
14. Mike is 16 years old. His father is 40 years old. What is the ratio of his father's age to Mike's?
15. There are 172 pupils in the ninth grade at the Adams High School. Of these, 43 are on the honor roll. What fraction of the ninth grade pupils are on the honor roll?
16. The basketball team at Adams High School has played 16 games and won 10 of them. What fraction of its games has it won?
17. Mr. Hansen and his family started at 9:00 A.M. on an automobile trip of 420 miles. By noon they had covered 168 miles. What fraction of their trip was left?
18. Last summer George earned \$240 and Eric earned \$210. What was the ratio of George's earnings to Eric's earnings?
19. What was the ratio of Eric's earnings to George's earnings?
20. Last year Miss Brown had 36 pupils in her class. This year she has 32 pupils in her class. The number in her class this year is what fraction of the number last year?
21. A water tower measures 96 feet high and 16 feet in diameter. What is the ratio of the height to the measure of the diameter?
22. What is the ratio of the measure of the diameter of the tower to its height?
23. Mr. Brown has an annual income of \$7500. His rent for one year amounts to \$1500. What fraction of his annual income is used for rent?

A statement that two fractions name the same ratio is called a *proportion*. This, for example, is a proportion:

$$\frac{3}{5} = \frac{9}{15}$$

Does this look familiar? In an earlier chapter we called $\frac{3}{5}$ and $\frac{9}{15}$ "equivalent fractions."

The numerators, 3 and 9, and denominators, 5 and 15, are the four *terms* of the proportion. The first numerator, 3, and the second denominator, 15, are called the *extremes*. The first denominator, 5, and the second numerator, 9, are the *means* of the proportion. Thus the means and extremes in the proportion above can be indicated in this way:

$$\begin{array}{r} \text{extreme} \rightarrow \\ \text{mean} \rightarrow \end{array} \frac{3}{5} = \frac{9}{15} \begin{array}{l} \leftarrow \text{mean} \\ \leftarrow \text{extreme} \end{array}$$

It is important to remember these names because of this useful property of all proportions.

The product of the means is equal to the product of the extremes.

In the proportion above, for example,

$$9 \times 5 = 45 \qquad \text{and} \qquad 3 \times 15 = 45$$

You can utilize this property to check the accuracy of any proportion. If the products are equal, it is a true statement.

1. Which of the following are true statements?

a. $\frac{3}{4} = \frac{9}{16}$ **b.** $\frac{5}{8} = \frac{20}{32}$ **c.** $\frac{6}{16} = \frac{18}{48}$ **d.** $\frac{15}{3} = \frac{60}{12}$

There is a simple explanation for the above property. If two equal fractional numbers are multiplied by the same number the products are equal. By taking the products of the means and extremes, the same result is obtained as if both fractional numbers were multiplied by the product of both denominators.

Thus, given the proportion:

$$\frac{7}{8} = \frac{21}{24}$$

the product of the means and the product of the extremes are the same:

$$7 \times 24 = 168 \qquad \text{and} \qquad 8 \times 21 = 168$$

You will get the same result if you multiply each fractional number by the product of the denominators, $8 \times 24 = 192$:

$$192 \times \frac{7}{8} = 168 \qquad \text{and} \qquad 192 \times \frac{21}{24} = 168$$

When one of the terms of a proportion is to be found, the proportion becomes a conditional statement. The value of the missing term is found by using the property of equality:

The product of the means equals the product of the extremes.

EXAMPLE

Solve: $\frac{N}{5} = \frac{14}{35}$

Taking the products of the means and extremes:

$35 \times N = 70$ or $35N = 70$ $\quad x \times y = p$

$N = 70 \div 35$ $\quad y = p \div x$

Then $N = 2$

2. Find the value of N in each of the following:

a. $\frac{N}{6} = \frac{8}{24}$

b. $\frac{15}{5} = \frac{N}{25}$

c. $\frac{1}{16} = \frac{5}{N}$

d. $\frac{N}{16} = \frac{15}{80}$

e. $\frac{4}{N} = \frac{12}{15}$

f. $\frac{N}{15} = \frac{12}{3}$

g. $\frac{45}{N} = \frac{21}{7}$

h. $\frac{5}{8} = \frac{10}{N}$

i. $\frac{3}{32} = \frac{9}{N}$

j. $\frac{4}{9} = \frac{16}{N}$

k. $\frac{18}{32} = \frac{27}{N}$

l. $\frac{2}{N} = \frac{3}{21}$

m. $\frac{3}{16} = \frac{N}{48}$

n. $\frac{N}{16} = \frac{30}{96}$

o. $\frac{2}{3} = \frac{N}{15}$

p. $\frac{N}{25} = \frac{7}{5}$

q. $\frac{75}{N} = \frac{36}{12}$

r. $\frac{3}{5} = \frac{9}{N}$

s. $\frac{7}{32} = \frac{N}{96}$

t. $\frac{5}{48} = \frac{25}{N}$

u. $\frac{2}{5} = \frac{N}{10}$

v. $\frac{N}{3} = \frac{8}{12}$

w. $\frac{N}{6} = \frac{4}{3}$

x. $\frac{5}{8} = \frac{20}{N}$

y. $\frac{9}{4} = \frac{N}{16}$

z. $\frac{25}{N} = \frac{5}{7}$

a′. $\frac{2}{3} = \frac{N}{30}$

b′. $\frac{3}{5} = \frac{N}{45}$

c′. $\frac{25}{N} = \frac{5}{3}$

d′. $\frac{3}{8} = \frac{N}{8}$

e′. $\frac{12}{36} = \frac{N}{15}$

f′. $\frac{3}{4} = \frac{N}{12}$

g′. $\frac{6}{7} = \frac{N}{7}$

h′. $\frac{N}{45} = \frac{2}{3}$

i′. $\frac{5}{8} = \frac{40}{N}$

j′. $\frac{15}{3} = \frac{N}{6}$

Rates are encountered in a variety of situations. You are familiar with rates such as the following:

(a) 4 cans for 75¢
(b) 100 yards in 9.6 seconds
(c) 80 miles on 15 gallons

Each of the rates may be expressed by a ratio; as

(a) $\frac{4}{75}$ (b) $\frac{100}{9.6}$ (c) $\frac{80}{15}$

Ratios are not read as fractions. The bar that separates the terms of a ratio is read as "to." Thus, $\frac{4}{75}$ is read "4 to 75."

Proportions are useful in solving problems about *rates*. For example:

A car travels 138 miles in 3 hours. At that rate, how far will it travel in 5 hours?

To find the answer, you could find how far the car travels in 1 hour, and multiply this answer by 5. A more economical way is to express the rate as a ratio, and set up a proportion, in order to find an equivalent ratio with denominator 5. Since the unit of measure of the first denominator, 3, is hours, then the second denominator, 5, must also be in hours. Similarly, since the first numerator, 138, is in miles, then the second numerator (the unknown) must also be in miles. Thus:

$$\frac{138}{3} = \frac{n}{5}$$

Equating the products of means and extremes,

$$\begin{aligned} 3n &= 690 & x \times y &= p \\ n &= 690 \div 3 & y &= p \div x \\ \text{then} \quad n &= 230 \end{aligned}$$

1. Express the ratio that represents each of the following rates.

a. 1750 revolutions in 5 seconds
b. 3 cans for $1
c. 76 miles on 5 gallons
d. 2 dozen eggs for 81¢
e. 1600 gallons in 5 minutes
f. 800 miles in 15 hours

2. Canned milk is advertised at 8 cans for $1. At that price, what is the cost of 30 cans?

3. A swimming pool is being filled with water that is flowing at the rate of 85 gallons in 2 minutes. The capacity of the pool is 7650 gallons. How long will it take to fill the pool?
4. Jim was paid $6.15 for selling 410 newspapers. How much would he have earned at this rate for selling 600 newspapers?
5. One and one-half tons of oranges sold at the packing plant for $90. At that rate, what is a 60-pound sack of oranges worth?
6. A contractor charged $1350 for pouring a concrete foundation containing 900 cubic feet. At that rate what should he charge for a foundation that will contain 1500 cubic feet of concrete?
7. An automatic automobile wash handled 25 cars in 1 hour and 15 minutes. How many could it handle at that rate in an 8-hour day?
8. A plane travels 1200 miles in $2\frac{1}{2}$ hours. At that rate how long should it take to travel 1800 miles?
9. A car used 3.5 gallons of gasoline in traveling 56 miles. At that rate how many gallons will it use on a trip of 800 miles?
10. A scout troop made a hike of 14 miles in 4 hours. At that rate how many miles should the troop travel in 6 hours?
11. A wheel made 1750 revolutions in 5 seconds. At that rate how many revolutions will the wheel make in one hour?
12. A jet plane traveled 2400 miles in 4 hours. At that rate how far should it travel in 6 hours?
13. Last summer the Gray family made a trip of 2500 miles in 5 days. At that rate how far did the family travel in 15 days?
14. Eleanor can buy 2 pounds of coffee for $1.38. At that rate how much will Eleanor have to pay for 6 pounds of coffee?
15. The speed of a radio signal from the moon travels at the rate of approximately 186,000 miles per second. If the moon is approximately 240,000 miles from earth, how long will it take the signal to be heard on earth?
16. A certain stock is listed at $50.50 per share. At that rate how many shares of this stock can Mr. King buy for $1000?
17. On a salary of $250, $10.50 was deducted for social security. At that rate what would be the deduction on a salary of $300?
18. On a trip of 1050 miles the Smith family used 70 gallons of gasoline. At that rate how many gallons of gasoline would the family use on a trip of 1575 miles?
19. A family with an annual income of $4800 spent $1200 for food. At that rate how much would a family with an annual income of $7200 spend for food?

In a recent football game between the Rams and the Colts, the quarterback for the Rams attempted 20 passes and completed 11 of them. The quarterback for the Colts attempted 25 passes and completed 13 of them. Which one completed the greater fraction of his passes?

Expressed as a fraction, the ratios of the number of passes completed to the number of passes attempted are: Rams: $\frac{11}{20}$ Colts: $\frac{13}{25}$

The ratios are more readily compared if they are expressed as decimals. To rename a fraction as a decimal, you need only remember that the fraction expresses division. That is, $\frac{11}{20}$ means $11 \div 20$.

Then, $\frac{11}{20} = 20\overset{.55}{\overline{)11.00}}$ and $\frac{13}{25} = 25\overset{.52}{\overline{)13.00}}$

Then the ratios expressed as decimals are: Rams: 0.55 Colts: 0.52

You can see that the Rams' quarterback had the higher ratio of completed passes. Because of their convenience in comparing ratios, decimals are used extensively for comparisons.

1. In the basketball game last night, Jim attempted 16 shots from the floor and sank 6 of them. Express the ratio of the number of shots made to the number of shots attempted as a decimal.

2. During the football season the four high schools in the city each played 16 games. The number of games won by each school was: Madison, 14; Jefferson, 12; Roosevelt, 10; and Lincoln, 13. Calculate, to the nearest thousandth, the ratio of the number of games won to the number of games played. Which team had the highest ratio? Which had the lowest?

3. In each pair of numbers named at the right the first is being compared to the second. Which number is the denominator for each pair?

4. When the denominator is greater than the number compared to it, the ratio will be less than 1. In which pairs is the ratio less than 1?

5. When the denominator is less than the number compared to it, the ratio is greater than 1. In which pairs will the ratio be greater than 1?

6. What is the ratio if the denominator is equal to the number compared to it? Is there a pair of numbers such that the ratio of one number to the other is 1?

a. 51 to 3
b. 30 to 50
c. 15 to 40
d. 8 to 56
e. 46 to 8
f. 33 to 60
g. 20 to 16
h. 15 to 10
i. 8 to 20
j. 25 to 5

7. Express the ratio for each pair of numbers as a fraction in lowest terms and as a decimal rounded to the nearest thousandth.

Last week Harry had 18 problems correct on a test of 20 problems. This week he had 22 problems correct on a test of 25 problems. On which test was the ratio of the number correct to the total higher?

To compare the ratios, let us express them in three different ways and see which provides the most convenient comparison:

(1) As a fraction the ratios are:

Last week: $\frac{18}{20}$ This week: $\frac{22}{25}$

(2) To express the ratios as decimals, you remember that we divide the numerator of the fraction by the denominator:

$$\text{Last week: } \frac{18}{20} = 20\overline{)18.00} \quad (.90)$$

$$\text{This week: } \frac{22}{25} = 25\overline{)22.00} \quad (.88)$$

Note that the ratios are readily compared when expressed as decimals.

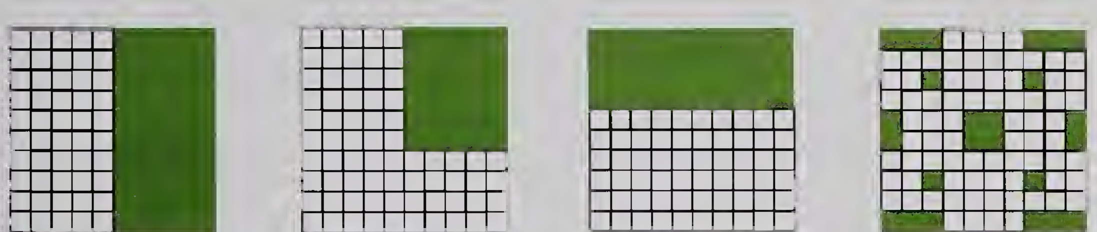
(3) To express the ratios as *per cents* we move the decimal point in each decimal two places to the right, and affix the per cent sign.

Last week: 90% This week: 88%

Per cent has the same advantages in providing convenient comparisons as has the decimal.

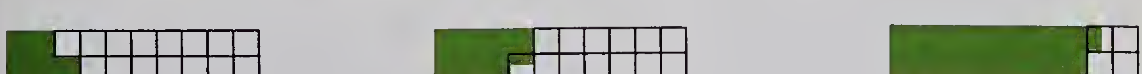
Per cent comes from a Latin expression that means per 100. The symbol % means "$\times \frac{1}{100}$" or "$\times$ .01." This explains why, when rewriting a decimal as a per cent, we move the decimal point two places to the right, and why we can write the per cent directly if we have a fraction whose denominator is 100.

1. Express in each of the 3 ways what part of each square region is shaded in the Figures below.



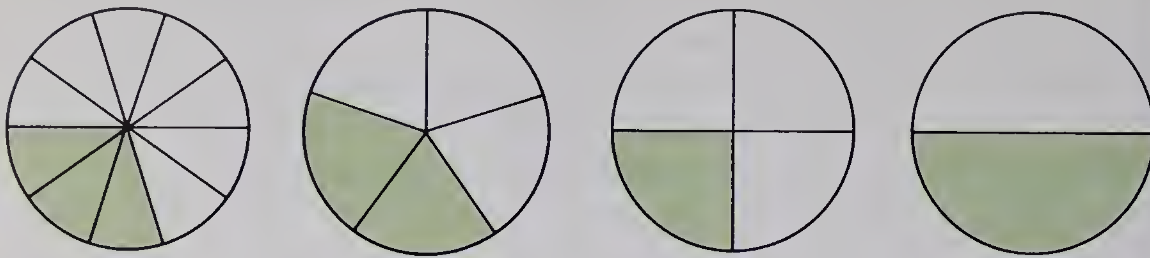
a. b. c. d.

2. Express in 3 ways what part of each rectangular region is shaded in the Figures below.



a. b. c.

3. Express in 3 ways what part of each circular region is shaded in the Figures below.



a. b. c. d.

Express the answer to each of the following problems in 3 ways.

4. The president of the music club announced that of the 60 members, 54 have paid their dues. What is the ratio of the number who have paid their dues to the total number of members?

5. Jim has been earning \$1.25 an hour. He has just gotten a raise of 5¢ an hour. This is what part of his original salary?

6. On a test of 16 questions Mabel had 10 correct. What is the ratio of the number of problems correct to the total number of problems?

7. Jane purchased a book that had been regularly priced at \$3.50 for \$2.80. The reduction was what part of the regular price?

8. Of a class of 32 pupils 4 were absent on Monday. This is what part of the class?

9. Last summer Mary went to Centralia to visit her cousin. She traveled 480 miles, 400 miles by plane and the rest by bus.

   a. What was the ratio of the distance traveled by plane to the distance traveled by bus?

   b. What was the ratio of the distance traveled by plane to the total distance?

10. On an assignment of 20 problems Jerry had 16 correct.

    a. What was the ratio of the number of problems correct to the total number of problems in the assignment?

    b. What was the ratio of the number correct to the number incorrect?

    c. What was the ratio of the number incorrect to the number of problems in the assignment?

11. During an after-Christmas sale Jane purchased a ball-point pen, regularly priced at 25¢, for 16¢.

    a. What was the ratio of the sale price to the regular price?

    b. What was the ratio of the discount to the regular price?

Per Cent

A. Write each of the following as a per cent.

1. 0.16 **3.** 1.47 **5.** 3.05 **7.** 0.608 **9.** 0.02
2. 0.09 **4.** 2.5 **6.** 4 **8.** 0.075 **10.** 0.005

B. Write each of the following as a per cent, rounded to the nearest tenth of 1% if there is a remainder.

1. $\frac{3}{4}$ **3.** $\frac{7}{10}$ **5.** $\frac{6}{11}$ **7.** $\frac{4}{15}$ **9.** $1\frac{1}{5}$
2. $\frac{5}{8}$ **4.** $\frac{3}{16}$ **6.** $\frac{5}{9}$ **8.** $1\frac{3}{4}$ **10.** $\frac{1}{200}$

C. Express the ratio of the first number named to the second number named in three ways:

a. As a fraction in lowest terms

b. As a decimal, rounded to the nearest thousandth if there is a remainder

c. As a per cent, rounded to the nearest tenth of 1%

1. 4 to 16 **5.** 9 to 5 **9.** 36 to 18 **13.** 30 to 40
2. 5 to 8 **6.** 36 to 40 **10.** 15 to 25 **14.** 40 to 25
3. 10 to 5 **7.** 18 to 4 **11.** 6 to 8 **15.** 18 to 6
4. 15 to 10 **8.** 25 to 10 **12.** 14 to 28 **16.** 7 to 5

D. Write each of the following as a decimal.

1. 57% **3.** 88.5% **5.** 4.4% **7.** 52.25%
2. 3% **4.** 0.8% **6.** 275% **8.** 112.8%

E. Find the value of n in each of the following:

1. 14 is n% of 35 **7.** 24 is n% of 30
2. n% of 600 is 3 **8.** 35 is n% of 25
3. 48 is n% of 40 **9.** n% of 500 is 498
4. 5 is n% of 8 **10.** n% of 20 is 30
5. 6 is n% of 24 **11.** 16 is n% of 20
6. n% of 12 is 9 **12.** n% of 25 is 75

If you need more practice, turn to the Practice Exercises on the following pages. If not, you may work in the Experts' Corner on page 223.

A. Writing a decimal as a per cent

Rule: To write a decimal as a per cent, move the decimal point two places to the right, and attach the per cent sign.

Write each of the following as a per cent.

1. .53	**7.** .382	**13.** .087	**19.** 1.025
2. .47	**8.** .837	**14.** .505	**20.** .675
3. .03	**9.** 1.09	**15.** .093	**21.** .045
4. .77	**10.** 3.30	**16.** 1.427	**22.** 1.11
5. .605	**11.** 4.25	**17.** 6.751	**23.** 2.35
6. .751	**12.** 6.525	**18.** 3.125	**24.** 5.0

B. Writing a fraction as a per cent

Rule: To write a fraction as a per cent, first write the fraction as a decimal, rounded to the nearest thousandth. Then write the decimal as a per cent to the nearest tenth of 1%.

If the fraction has a denominator of 100, then (by definition of per cent), we can write the per cent directly. For example, $\frac{3}{100} = 3\%$, $\frac{18.5}{100} = 18.5\%$, etc.

The following Example illustrates the use of a proportion in writing a fraction as a per cent.

EXAMPLE

$\frac{4}{7} = n\%$ Using the definition of per cent, we can set up the proportion:

$$\frac{4}{7} = \frac{n}{100}$$

Equating the products of the means and extremes:

$$7n = 400 \qquad x \times y = p$$
$$n = 400 \div 7 \qquad y = p \div x$$
$$n = 57.1 \text{ (to the nearest tenth)}$$

Then $\frac{4}{7} = \frac{57.1}{100}$

and $\frac{4}{7} = 57.1\%$

Write each of the following as a per cent, rounded to the nearest tenth of 1%. You may use the above rule or the proportion method illustrated in the Example.

1. $\frac{3}{4}$	**6.** $\frac{4}{5}$	**11.** $\frac{3}{11}$
2. $\frac{5}{6}$	**7.** $\frac{5}{7}$	**12.** $\frac{5}{32}$
3. $\frac{2}{7}$	**8.** $\frac{9}{16}$	**13.** $\frac{4}{17}$
4. $\frac{4}{15}$	**9.** $\frac{1}{16}$	**14.** $\frac{3}{31}$
5. $\frac{3}{7}$	**10.** $\frac{7}{29}$	**15.** $\frac{5}{19}$

C. Expressing the ratio of one number to another

To express the ratio of one number to another as a fraction, identify the denominator. The number being compared to the denominator is the numerator. To express the ratio as a decimal, use the denominator as the divisor. The number being compared to the denominator is the dividend.

To express the ratio as a per cent, first express the ratio as a decimal, and change the decimal to a per cent.

Unless directed otherwise, round decimal calculations to the nearest thousandth, and per cent to the nearest tenth of 1%, if there is a remainder.

EXAMPLE

Express in three ways the ratio of 35 to 56.

As a fraction, the ratio is $\frac{35}{56}$, or $\frac{5}{8}$.
As a decimal: $\frac{5}{8} = .625$
As a per cent: $.625 = 62.5\%$

Express the ratio of the first number to the second in each of three ways.

1. 4 to 16	**10.** 16 to 50	**19.** 4 to 200
2. 8 to 10	**11.** 19 to 32	**20.** 17 to 80
3. 36 to 18	**12.** 15 to 21	**21.** 25 to 10
4. 5 to 20	**13.** 175 to 200	**22.** 56 to 14
5. 25 to 20	**14.** 16 to 5	**23.** 21 to 7
6. 12 to 40	**15.** 19 to 40	**24.** 17 to 68
7. 68 to 16	**16.** 240 to 160	**25.** 25 to 5
8. 64 to 10	**17.** 13 to 20	**26.** 20 to 16
9. 50 to 40	**18.** 42 to 70	**27.** 16 to 64

D. Writing a per cent as a decimal

Rule: To write a per cent as a decimal, remove the per cent sign and move the decimal point two places to the left.

Write these per cents as decimals.

1. 56%	**4.** 95%	**7.** 0.83%	**10.** 9%	**13.** 9.5%
2. 5%	**5.** 13.2%	**8.** 6.3%	**11.** .015%	**14.** 2%
3. 120%	**6.** 150%	**9.** .07%	**12.** 103.4%	**15.** 1.25%

E. Finding what per cent one number is of another

Rule: To find what per cent one number is of another: (1) express the ratio as a fraction in simplest form; (2) write the fraction as a decimal rounded to the nearest thousandth; (3) write the decimal as a per cent to the nearest tenth of 1%.

Another procedure is to use a proportion, as the following Examples illustrate.

EXAMPLES

1. 10 is $n\%$ of 16

Since $n\% = \frac{n}{100}$,

$$\frac{n}{100} = \frac{10}{16}$$

$16n = 1000$ $\qquad x \times y = p$

$n = 1000 \div 16$ $\qquad y = p \div x$

$n = 62.5$

Then 10 is 62.5% of 16.

2. 37 is $n\%$ of 21

Since $n\% = \frac{n}{100}$,

$$\frac{n}{100} = \frac{37}{21}$$

$21n = 3700$ $\qquad x \times y = p$

$n = 3700 \div 21$ $\qquad y = p \div x$

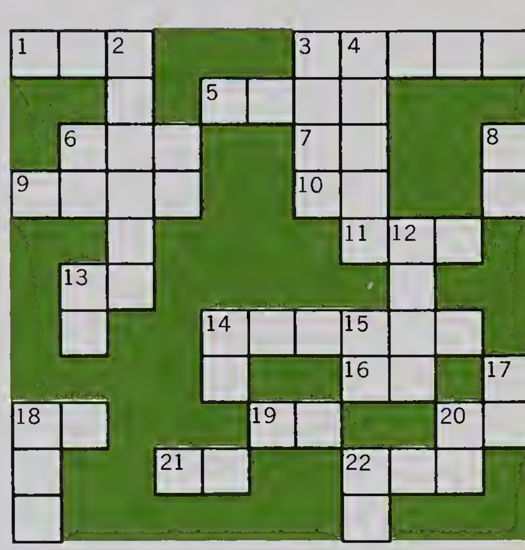
$n = 176.19$

Then 37 is 176.2% of 21.

Solve each of the following for N.

1. 18 is $N\%$ of 45	**7.** 63 is $N\%$ of 140	**13.** 28 is $N\%$ of 32
2. $N\%$ of 48 is 32	**8.** 23 is $N\%$ of 75	**14.** 5 is $N\%$ of 33
3. 17 is $N\%$ of 136	**9.** $N\%$ of 35 is 15	**15.** 17 is $N\%$ of 23
4. 27 is $N\%$ of 40	**10.** 6 is $N\%$ of 24	**16.** 85 is $N\%$ of 70
5. 52 is $N\%$ of 65	**11.** 2 is $N\%$ of 400	**17.** 3 is $N\%$ of 439
6. 16 is $N\%$ of 25	**12.** 30 is $N\%$ of 20	**18.** 3 is $N\%$ of 70

First copy the given Figure. Then to solve the puzzle, perform the operations indicated below and insert the answers into the corresponding blanks in the Figure.



ACROSS:

1. Add 443 and 214.
3. Write as a numeral: twenty-three thousand six hundred forty-six.
5. Multiply 46 by 58.
6. Divide 1945 by 5.
7. Divide 3388 by 44.
9. 831 + 635 + 987
10. Add 2 to the product of 7 and 12.
11. Multiply 18 by itself.
13. Subtract 11 from the product of 14 and 6.
14. Round 456,789 to the nearest hundred.
16. How many tens are in 500?
17. Write the number symbol for the smallest natural number.
18. Divide 2158 by 26.
19. Write the numeral for the greatest odd number named by two digits.
20. Find the sum of the numbers named by the digits in 284,637.
21. Divide 121 by 11.
22. Write the numeral for the next odd number greater than 525.

DOWN:

2. Write as a numeral; seven hundred ninety-eight thousand five hundred sixty-three.
3. Subtract 3562 from 6240.
4. Multiply 657 by 59.
6. Subtract 3562 from 3596.
8. Write a two digit numeral in which the sum of the numbers named by the digits is 12 and the number named by the units digit is three times that of the tens digit.
12. Round 2645 to the nearest hundred.
13. Divide 6786 by 87.
14. $2^4 \times 3 = ?$
15. Four plus nine times nine
18. Using a single digit, write a three digit numeral in which the sum of the numbers named by the digits is 24.
20. $3 + (17 \times 2)$
22. 2×5^2

In a newspaper story at the end of the football season, this summary appeared comparing the performances of two quarterbacks. The following is a statistical comparison of two of the league's quarterbacks, Harry Jones and John Compton.

TOTAL FOR SIX GAMES

	Compton	*Jones*
Passes attempted	127	101
Passes completed	94	59
Passes intercepted	4	5
Yards gained	1022	687
Touchdown passes	10	4

1. What is meant by "passes attempted" and "passes completed"?

2. Which quarterback completed more passes? Which quarterback gained more yards?

3. How many passes did John Compton attempt? How many passes did he complete?

4. One measure of effectiveness of a quarterback is the ratio of the number of passes completed to the number of passes attempted. What fraction of his passes did Compton complete? Express this as a fraction and as a decimal rounded to the nearest thousandth.

5. The newspaper account states that Compton had completed approximately 74% of his passes. As you know, this means .74 of his passes. Round your decimal for Exercise 4 to hundredths and see if this is correct.

6. What fraction of his passes did Jones complete? Express this as a fraction and as a decimal rounded to the nearest thousandth.

7. Round your answer in Exercise 6 to the nearest hundredth, and state what per cent of his passes Jones completed.

8. Which of the quarterbacks completed the greater per cent of his passes?

9. What per cent of Compton's completed passes were touchdown passes? How about Jones?

10. Which of the two quarterbacks threw the greater per cent of touchdown passes?

11. Jim Harris, quarterback for the Central High School team, attempted 16 passes in the game last week, and completed 6 of them. What per cent of his passes did he complete? Express the ratio also as a decimal and as a fraction.

DIVIDE OR SUBTRACT?

In each of the following exercises you are asked to make a comparison between numbers. You are to decide whether to make the comparison by division or by subtraction. List the numerals 1 through 15 on a sheet of paper. After reading each exercise write D or S after the numeral to indicate whether you will make the comparison by division or by subtraction. Then perform the computation, and write the answer.

1. The president of the Camera Club announced that of the 47 members, 13 had not yet paid their dues. How many members had paid their dues?
2. The Washington Monument is 555 feet high. The Statue of Liberty is 250 feet high. The former is how many feet higher than the latter?
3. The Washington Monument is how many times as high as the Statue of Liberty?
4. Ethel paid \$27.50 for a camera that regularly sells for \$40. How much did she save from the regular price?
5. What per cent of the regular price did Ethel save?
6. Margaret received her weekly allowance of \$7.50 last Monday. On Friday she still had \$1.15 left. How much had she spent?
7. In 1950 the population of Glendale was 52,000. In 1960 its population was 65,000. How much was the increase in population?
8. The increase in population in Glendale was what per cent of its 1950 population?
9. The enrollment of Jefferson High School is 1200. The enrollment of Emerson High School is 1500. Express in 3 ways the ratio of the Jefferson enrollment to that of Emerson.
10. Express in three ways the ratio of enrollment in the Emerson High School to that in Jefferson High School.
11. The enrollment in Emerson High School is how much greater than that in Jefferson High School?
12. John is $66\frac{1}{2}$ inches tall. Henry is $64\frac{1}{4}$ inches tall. John is how much taller than Henry?
13. Mr. Brown and his family started on an auto trip at 10:00 A.M. At noon they had gone 102 miles, and had 203 miles left to travel. What per cent of the total distance had they traveled? Round your answer to the nearest tenth of one per cent.
14. Explain how you can tell, in making a comparison, whether you should subtract or divide.

The use of a proportion as a conditional statement provides an economical method of solving many problems in which quantities are compared, as well as where rates are involved.

EXAMPLE

In a recent basketball game Henry shot 35 times from the floor and made 15 baskets. What per cent of his shots were successful?

The ratio of the number of successful shots to total attempts is $\frac{15}{35}$. To express this as a per cent we solve the proportion:

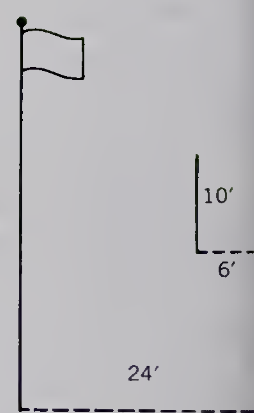
$$\frac{15}{35} = \frac{n}{100}$$
$$15 \times 100 = 35 \times n$$
$$1500 = 35n$$
$$42.86 = n$$

The ratio, rounded to the nearest tenth of one per cent, is 42.9%.

The general procedure for using proportions in problem solving is to find two expressions for the same ratio, one of which has a missing term. The proportion is then set up as the conditional statement.

EXAMPLE

At any time of day the ratios of the heights of two objects to their shadows are equal. When a flagpole casts a shadow 24 feet long, a 10-foot fence post casts a shadow 6 feet long. How high is the flagpole?



Since: $\frac{\text{height of flagpole}}{\text{its shadow}} = \frac{\text{height of fence post}}{\text{its shadow}}$

Then: $\frac{n}{24} = \frac{10}{6}$

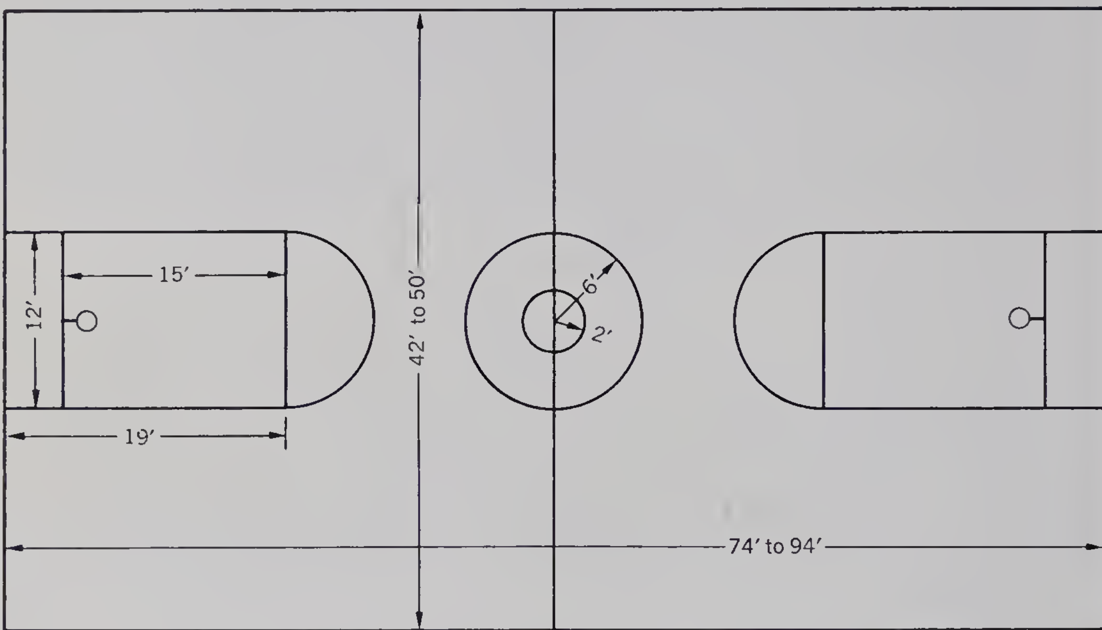
Complete the solution.

Set up the proportion and solve each of the following problems.

1. Jane paid \$3 for a book regularly priced at \$3.50. What per cent of the regular price did she save?
2. Mike plans to save 45¢ out of each dollar he earns next summer. If he earns \$280 how much should he save?

3. In 1950 Plainville had a population of 2400 persons. In 1960 the population was 2640. The increase was what per cent of the 1950 population?
4. A tree casts a shadow 25 feet long when a 6-foot fence post casts a shadow 5 feet long. How high is the tree?
5. If Mr. Adams plows 24 acres in 10 hours, how long should it take him to plow an 80-acre field? Round your answer to the nearest tenth.
6. The taxes on a farm assessed at $14,400 are $276.38. At that rate what should be the taxes to the nearest cent on a farm assessed at $20,000?
7. During the first 5 days of the month a merchant's sales were $8750. At that rate, how much should he expect the sales to amount to in a month of 26 business days?
8. If a 30-foot lot sells for $2500, what should a 75-foot lot sell for, at the same rate?
9. Four yards of cloth are sold for $1.92. What should 9 yards of the same material sell for?
10. Henry purchased a catcher's mitt priced at $7.50. The sales tax was 30¢. This was what per cent of the price of the mitt?
11. Three cans of fruit juice sell for 25¢. What will $1\frac{1}{2}$ dozen cans sell for?
12. A tree casts a shadow 27 feet long when Helen, who is 5 feet tall, casts a 3-foot shadow. How high is the tree?
13. On a map a segment 3 inches long represents a distance of 100 miles. How many miles would be represented by a line segment 4 inches long? How would a distance of 500 miles be represented?
14. A road map has a scale of 4 inches to 500 miles. How many inches would represent a distance of 1250 miles?
15. A filling station sells gasoline for 32.9¢ a gallon. Of this amount 11¢ is for taxes. What per cent of the selling price is tax? Round your answer to the nearest tenth of one per cent.
16. A ship burns 3500 tons of fuel on a 12-day voyage. How much fuel (to the nearest ton) should be allowed for a 25-day voyage?
17. A plane flies 3000 miles in 8 hours. How long will it take to fly 1200 miles at the same rate?
18. For the first six days of his vacation Mr. Henderson's expenses were $70. At that rate, how long can his vacation last on $175?
19. A jet plane is scheduled to travel 560 miles per hour. A propeller-driven plane is scheduled to travel 350 miles per hour. This is what per cent of the distance per hour traveled by the jet plane?

Anyone who is planning to build a house, an article of furniture, lay out a baseball or basketball playing area, or design a real estate development needs a plan to follow. For this purpose a *scale drawing* is necessary. The *scale* tells what ratio the length of a segment on the drawing is to the same dimension on the object. Here is a scale drawing of a regulation basketball court.



1. While the dimensions of a basketball court may vary, the length may not be less than 74 feet, nor the width less than 42 feet. How can you tell this from the scale drawing?
2. The backboard to which the basket is attached is 15 feet from the center of the free throw circle. How is this indicated?
3. By measuring some of the dimensions on the drawing determine the scale of the drawing, and find the width and length of the court.
4. Sometimes the scale of the drawing is indicated by a ratio written with a colon instead of a fraction bar. An example of this is in the drawings of the butterfly. Write the scale of the drawing in Figure a below as a fraction.


SCALE 1:4
a.


SCALE 1:8
b.


SCALE 1:16
c.

5. The real butterfly is how many times as large as the drawing in Figure b?

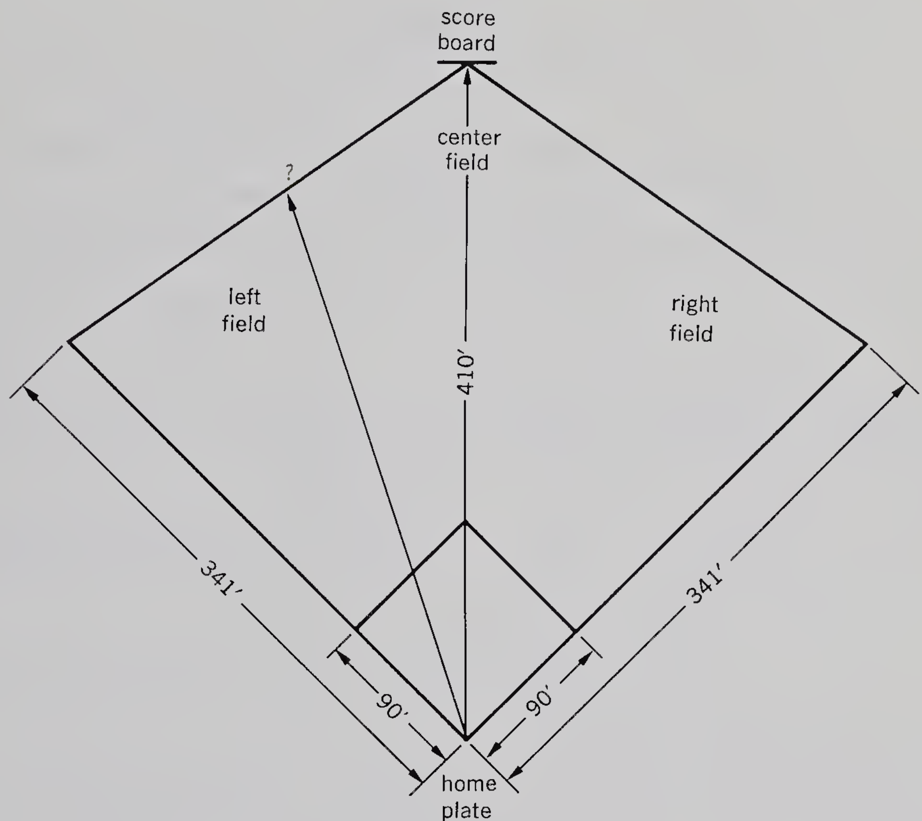
6. Write the scale used in the drawing of the basketball court, both as a fraction, and using a colon.

7. In a science book there is a picture of a very small insect, with the scale reading: *Enlarged 16 times.* In the drawing the insect is 2 inches long. What is the actual length of the insect?

8. Express the scale of the drawing in Exercise 7 using a colon.

9. The Figure below is the layout of the playing area of Shea Stadium, home of the New York Mets baseball team. The scale is $1'' = 150'$. Express this as a fraction and also using a colon.

10. The distance from home plate to first base is 90 feet. How long is this segment on the drawing? Measure it first, and then check your measurement by using a proportion.

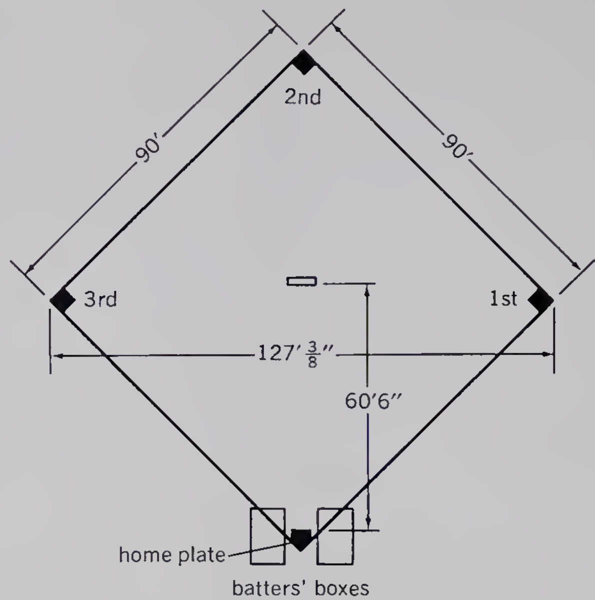


11. It is 410 feet to the scoreboard in center field from home plate. How long to the nearest $\frac{1}{8}$ inch, is the segment in the drawing that represents this?

12. A batter hits a home run that clears the fence in left field at the point indicated by a question mark (?). How far, to the nearest ten feet, is this point from home plate ?

13. While the distances to the fences at the foul lines and in center field differ in the various major league ball parks, the layout of the

infield is standardized in the regulations. This is shown in the Figure below. By making some measurements determine what scale was used in the drawing.



14. A batter gets a hit to center field and reaches second base in 8 seconds. How many yards per second does he run?

15. A catcher standing $4'\frac{5}{8}''$ back of home plate throws to second base. It takes the ball one second to reach second base. How many miles per hour was the ball traveling?

16. On a wall map the distance from San Francisco to New York measures $34\frac{1}{2}''$. The scale reads "$1'' = 75$ miles." What is the actual distance from San Francisco to New York?

17. A scale commonly used in "working" drawings for furniture construction is $\frac{3}{16}'' = 1''$. Using this scale, what are the dimensions of the drawing of a table top $3'$ wide and $5'$ long?

18. The table is $30''$ high. What is the length of the segment in the drawing that represents this?

19. On a scale drawing the ratio of the measure of each segment to the measure of each segment it represents remains the same. What is the ratio of the measure of each segment to the segment it represents in drawings with each of these scales?

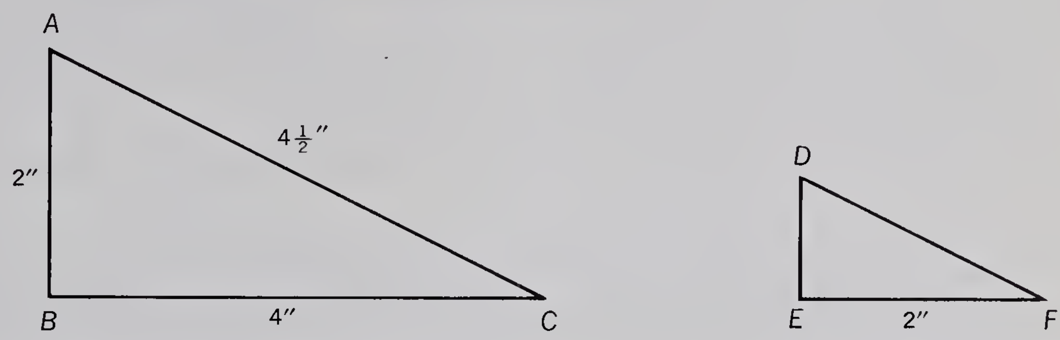
a. $3'' = 1'$ **b.** $1'' = 1'$ **c.** $\frac{3}{16}'' = 1'$ **d.** $\frac{1}{12}'' = 1'$

20. Draw a scale drawing of a table top that measures $6' \times 4'$. Use $1'' = 2'$.

21. The height of the table in Exercise 20 is 30 inches. Using the same scale as in Exercise 20, draw a scale drawing of the side view of the table.

Similar Figures

Two plane figures that have the same shape but not necessarily the same size are called *similar figures.* These two right triangles are similar. Note that they are drawn to scale.



1. The lengths of corresponding sides of similar figures have the same ratio. What is the ratio of EF to BC?
Remember! We use EF to mean "the measure of $\overline{EF}$." Similarly, BC means "the measure of $\overline{BC}$."

2. The ratio of DE to AB is the same, since the triangles are similar. What is the measure of $\overline{DE}$?

3. How long is $\overline{DF}$?

4. Construct a right triangle with the right angle at B with $AB = 1\frac{1}{2}''$, and $BC = 2''$. Construct another right triangle with the right angle at E, with $DE = 3''$, and $EF = 4''$.
Measure to see if this proportion is true: $\dfrac{AB}{DE} = \dfrac{BC}{EF}$.

5. Set up three other proportions that are also true.

6. An important property of similar figures is this:

In similar figures, the measures of corresponding angles are equal, and the measures of corresponding sides are in the same ratio.

Measure the angles of the two triangles you constructed in Exercise 4, and see if the triangles are similar.

7. The following is an important property of similar triangles:

Two triangles are similar if the measures of two angles of one are equal to the measures of two angles of the other.

Construct $\triangle ABC$ and $\triangle DEF$ with angles A and D each measuring $40°$, and angles B and E each measuring $50°$, with $AB = 2''$ and $DE = 4''$. Set up three proportions to show that the triangles are similar.

8. Verify this proportion: $\dfrac{AB}{AC} = \dfrac{DE}{DF}$

9. Set up two other true proportions similar to that in Exercise 8.

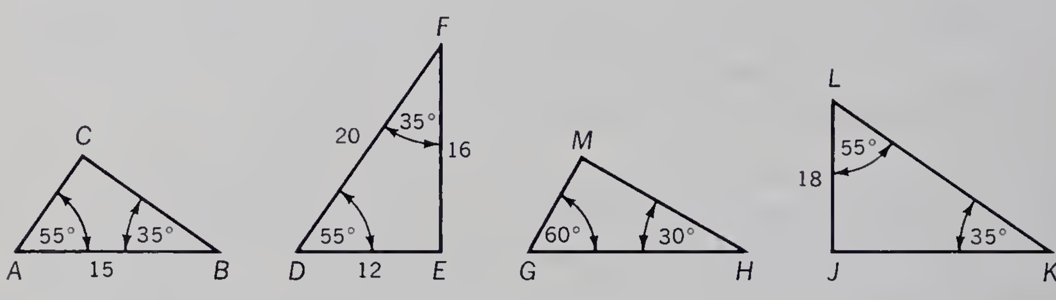
10. Construct a rectangle $3''$ long and $2''$ wide. Construct another similar to the first that is $4\frac{1}{2}''$ long. See how many true proportions you can set up comparing the measures of the sides.

11. The following is another important property of similar triangles:

> Two triangles are similar if the measures of two sides of one are proportional to those of two sides of the other, and the measures of the included angles are equal.

The "included angle" is the angle determined by the two sides. Construct $\triangle ABC$ with $AB = 2''$, $AC = 1\frac{1}{2}''$, and $m\angle A = 30°$. (Remember! We are using the shorthand "$m\angle A$" to mean "the measure of angle A.") Construct $\triangle DEF$ with $m\angle D = 30°$, $DE = 4''$, and $DF = 3''$. Set up as many true proportions as you can identify among the sides of the triangles.

12. In the Figure below, how do you know that $\triangle ABC$ is similar to $\triangle DEF$?



13. What is the ratio of the measure of any side of $\triangle ABC$ to that of the corresponding side of $\triangle DEF$?

14. Set up three proportions between the measures of the sides of the triangles ABC and DEF.

15. How long is side $\overline{AC}$? Side $\overline{BC}$?

16. Which of the triangles GHM and JKL in the Figure above is similar to $\triangle ABC$? Explain how you know.

17. Set up the proportions, and find the measures of each of the sides of the triangle you think is similar to $\triangle ABC$.

18. Construct a triangle similar to $\triangle XYZ$ (Figure 1 below), with each side measuring $1\frac{1}{2}$ times as long. How many sides of the triangle that you construct must you measure?

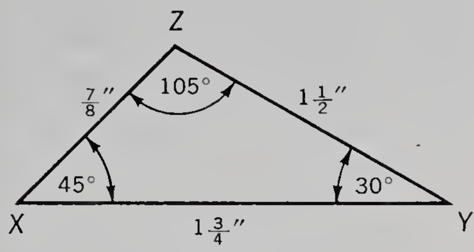


Figure 1

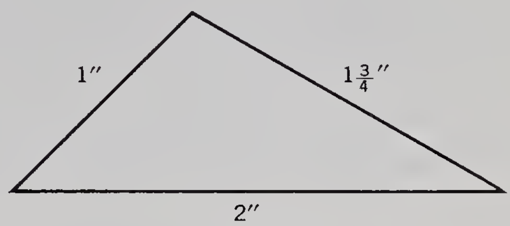


Figure 2

19. You may already have discovered the following property:

Two triangles are similar if the measures of the corresponding sides are proportional.

Construct a triangle similar to Figure 2 above, with each side measuring twice as long.

20. Construct triangle ABC with $m\angle A = 30°$, $m\angle B = 60°$, and $AB = 2$ inches. What is the measure of $\overline{BC}$? of $\overline{AC}$?

21. Construct triangle DEF with $m\angle D = 30°$, $m\angle E = 60°$, and $DE = 3$ inches. Is triangle DEF similar to triangle ABC? If so, state the rule that explains why.

22. Calculate, without measuring, the measures of $\overline{DF}$ and $\overline{EF}$. Then measure to verify the accuracy of your calculations.

23. Referring to the two triangles you just constructed, calculate the numerical value of each of the following ratios. State which proportions are true, and which are false.

a. $\frac{AC}{FD} = \frac{BC}{EF}$ **c.** $\frac{AC}{AB} = \frac{DF}{DE}$ **e.** $\frac{BC}{AB} = \frac{DF}{DE}$

b. $\frac{BC}{EF} = \frac{AB}{DE}$ **d.** $\frac{AC}{BC} = \frac{DF}{EF}$ **f.** $\frac{EF}{BC} = \frac{DE}{AB}$

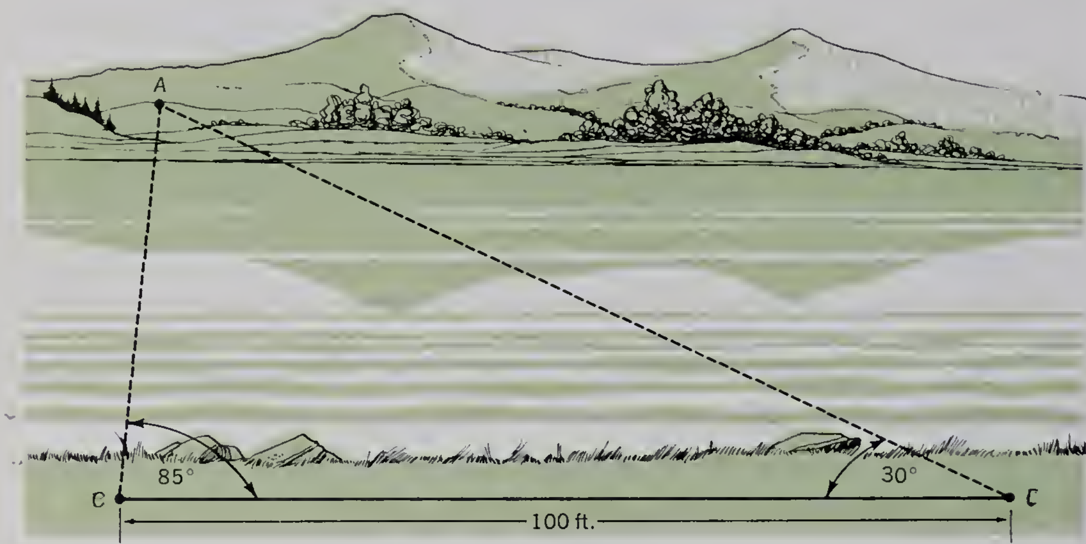
24. Construct triangle ABC with $AB = 3$ inches, $AC = 4$ inches, and $m\angle A = 90°$. What is the measure of $\overline{BC}$?

25. Construct triangle DEF with $DE = 1\frac{1}{2}$ inches, $DF = 2$ inches, and $m\angle D = 90°$. Is triangle DEF similar to triangle ABC? If so, state the rule that tells you the two triangles are similar.

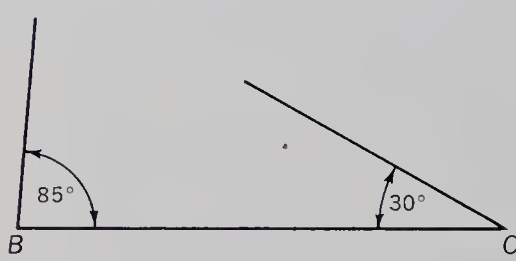
26. Calculate the measure of $\overline{EF}$, then verify it by measurement.

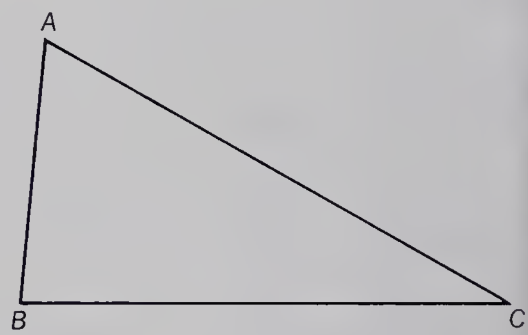
The Wilson High School Athletic Association needed to know the width of the river where they hold their annual swimming meet.

Richard Smith said he could do it by using a scale drawing, if he could use the *transit* to measure two angles. (See the Figure below.) He put up stakes at points A, B, and C. Each stake is 10 feet back from the river, and point C is 100 feet from B. He set up the transit at B. He found that the angle at B measured 85°. Setting up the transit at C he found that angle C measured 30°.



1. Richard used the scale $1'' = 20'$. How many inches long should he make $\overline{BC}$?





2. He laid off $m\angle B = 85°$ and $m\angle C = 30°$ with a protractor and then completed the triangle. (See the Figures above.) Measuring $\overline{AB}$ on his scale drawing he found it was $2\frac{3}{4}''$ or $2.75''$ long.

   Setting up the proportion $\frac{2.75}{5.0} = \frac{N}{100}$, Richard found the actual length of $\overline{AB}$. Solve for N, and state how many feet it is across the river from A to B.

3. Since the stakes are each 10 feet from the river at A and B, how much should Richard subtract from his measurement to find the width of the river?
4. How far is it across the river?
5. While a surveyor's transit is useful for precise work, you can make one for yourself that is sufficiently accurate for ordinary measurement. The one shown in Figure 1 was made by a high school class to measure the distance to an island in a lake (Figure 2).

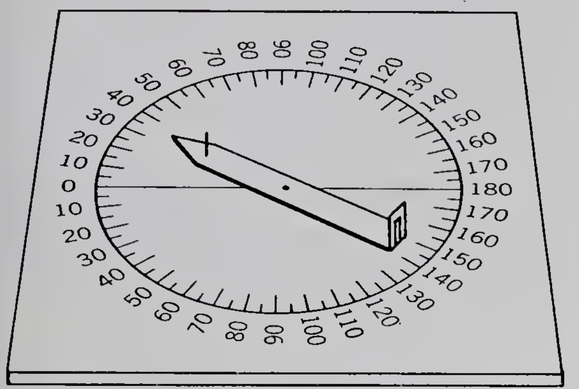


Figure 1

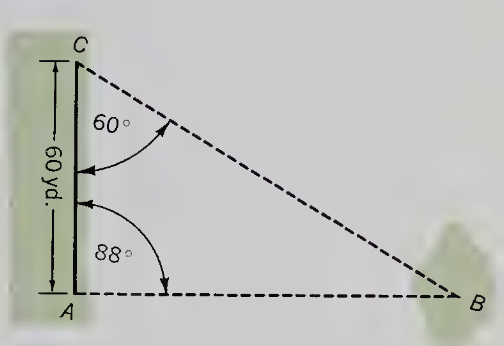
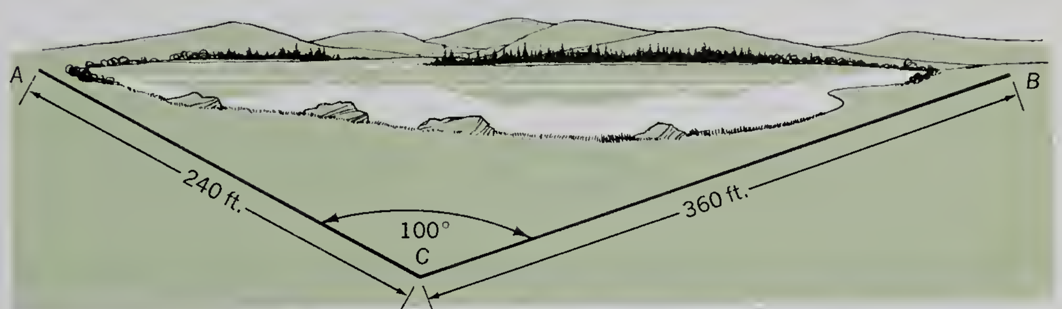


Figure 2

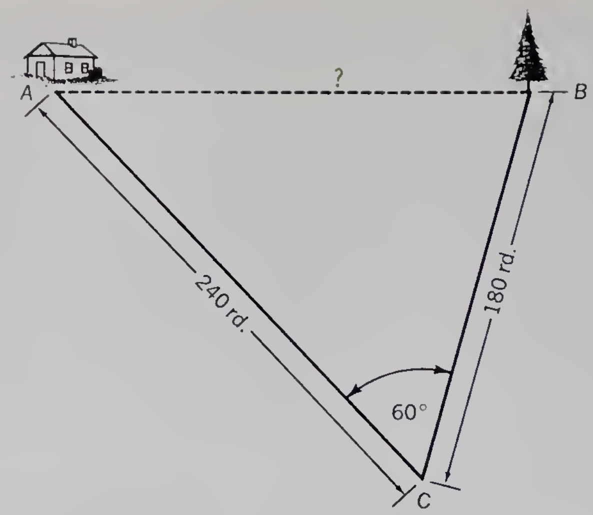
Make a scale drawing using 1 inch = 20 yards and find the distance from A to B.

6. The Boy Scouts made these measurements (in the Figure below) to determine the length of the lake at their summer camp.



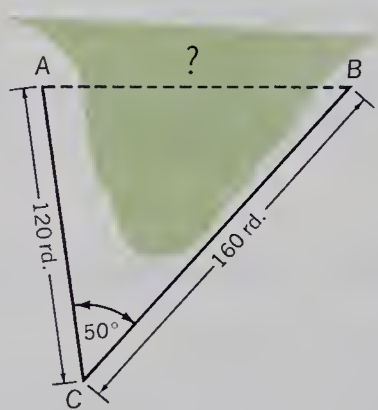
Make a drawing with the scale 1 inch = 80 feet, and find the distance from A to B.

7. To locate precisely the boundaries of his farm, Mr. Jenkins needed to know the distance from his house at point A to a tree at point B. Since there was a steep hill between these points he could not measure the distance directly. He could, however, measure the distance to both A and B from point C. He used the scale 1 inch = 60 rods and made a scale drawing. He completed the triangle by connecting points A and B. (See the Figure at the top of page 236.)

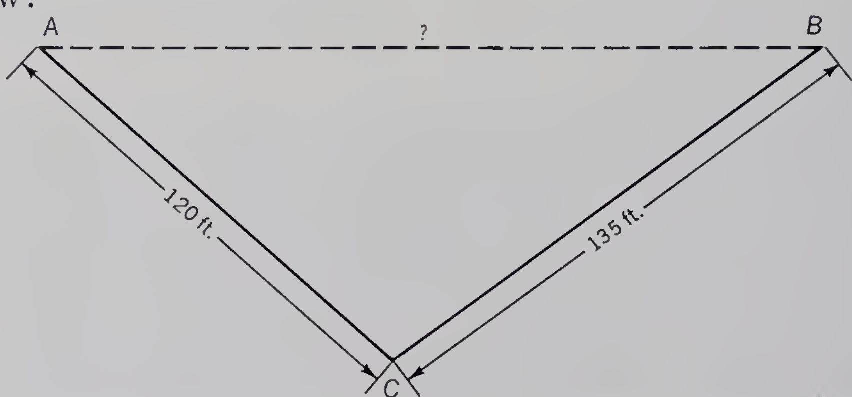


Make a scale drawing using 1 inch = 60 rods. (What is the measure of $\overline{AB}$ in your drawing? What is the scale?) Set up a proportion, and find the number of rods between his house at A and the tree at B.

8. Mr. Adams owns an ocean front lot with an inlet where he keeps his boat. Make a scale drawing with 1 inch = 40 rods. How far is it from A to B in the Figure below?



9. A waterpipe runs underground from A to B. (See the figure below.) How many feet of pipe are there if $AB = 3\frac{5}{16}''$ in the scale drawing below?



1. The basketball team has won 12 games and lost 3 games.

a. How many games has the team played?
b. What is the ratio of the number of games won to the number of games played expressed as a fraction? as a decimal?
c. What is the ratio expressed as a per cent?

2. Mary works as a secretary afternoons and Saturdays, 12 hours a week, at 90¢ an hour. She saves \$3.60 a week.

a. How much does Mary earn a week?
b. What per cent of her earnings does she save?
c. How much can she save in a year (52 weeks)?

3. A field is 60 rods wide. Its perimeter is 320 rods.

a. How many rods is the sum of the measures of the two shorter sides?
b. How many rods is the sum of the measures of the two longer sides?
c. What is the number of rods in the length of the field?
d. Express the ratio of the width to the length as a fraction.

4. The Lakeside basketball team has won 8 games and lost 4. The Hillcrest team has won 10 games and lost 6.

a. How many games has the Lakeside team played?
b. Express the ratio of the number of games won by Lakeside to the number of games played as a decimal.
c. How many games has the Hillcrest team played?
d. Express the ratio of the number of games won by Hillcrest to the number of games played as a decimal.
e. Which team has won the greater fraction of its games?

5. The ratio of the width to the length of a field is $\frac{5}{16}$. The width is 40 rods.

a. How many rods is $\frac{1}{16}$ of the length of the field?
b. What is the length of the field?
c. What is the ratio of the length of the field to its width?

6. John is 4′9″ tall. Mike is 5′ tall.

a. How many inches tall is John? Mike?
b. What is the ratio of John's height to Mike's height expressed as a fraction?
c. What is the ratio of Mike's height to John's height expressed as a fraction?

How well can you use the steps for solving applied problems?

STEPS FOR SOLVING APPLIED PROBLEMS

1. Understand the problem.
2. Note what the problem asks for.
3. Look for hidden questions.
4. Estimate a reasonable answer.
5. Set up and solve the conditional statement(s).
6. Check your answer.

1. Mr. Anderson earns \$5800 a year. His family spends \$1392 a year for food. What per cent of the income is spent for food?

2. In a class of 40 pupils, 2 pupils were absent last Monday. What part of the class was absent? Express this as a fraction; as a decimal; as a per cent.

3. On a test of 24 questions, George had 4 wrong. What per cent did he have correct?

4. Henry had been earning \$45 a week, and received a raise of \$8 a week. What per cent of his former pay is his increase?

5. Elizabeth bought a book that was regularly priced at \$3.50 for \$2.80. What per cent of the regular price did Elizabeth save?

6. In 1940 Lakata had a population of 2400. In 1950 the population was 2760. The increase was what per cent of the 1940 population?

7. The president of the camera club announced recently that of the 40 members, 36 had paid their dues. What per cent of the members have paid their dues? Therefore, what per cent of the members had not paid their dues?

8. On Monday morning Henry's father gave him \$2.50. On Saturday night he still had 35¢. What per cent of his money did he still have?

9. The Austin High School basketball team played 16 games and won 12 of them. The Avon High School basketball team played 10 games and won 8 of them. Which team won the greater per cent of its games?

10. Mr. Williams and Mr. Brown agreed to share the cost of a fence between their yards, with Mr. Williams paying \$363 and Mr. Brown paying the rest. If the fence cost \$484, how much did Mr. Brown pay? What per cent did Mr. Williams pay?

Puzzle problems usually deal with a trivial or improbable situation. Nevertheless they provide opportunities to practice the problem-solving steps for mathematical problems. See if you can apply each of them in solving these problems.

STEPS FOR SOLVING MATHEMATICAL PROBLEMS

1. Understand the problem.
2. Analyze the data.
3. Discover new facts.
4. Follow up and verify promising leads.
5. Review your solution.

1. A customer asked a jeweler to make a continuous chain of 20 gold links from 5 pieces each having 4 links. The jeweler said it would cost \$5, as he charges 50¢ to cut a link and 50¢ to solder it. The customer said it should cost \$4. Who was right?

2. Jim started out to go duck hunting one morning, walking at 3 miles an hour toward a lake some distance away. Half an hour later George showed up with his dog, and was told where Jim was going. George hurried after him, walking at a rate of $4\frac{1}{2}$ miles an hour. However, his dog set out at 10 miles an hour, caught up to Jim, and raced at the same rate back to George. How far had Jim gone when the dog caught him? How far had George gone when the dog got back?

3. A barrel is filled with apples in 1 hour by doubling the number of apples each minute. When was it half full?

4. Four monkeys eat 4 sacks of peanuts in 3 minutes. How many monkeys, at the same rate, will it take to eat 100 sacks of peanuts in 1 hour?

5. Mr. Adams, for reasons best known to himself, wishes to transport a fox, a goose, and a sack of corn across a river. The only boat available will carry only two at a time. For obvious reasons the goose cannot be left alone either with the corn or the fox. How can Mr. Adams manage to get all safely across?

6. At a sale two pianos were sold for \$720 each. The selling price of one was $\frac{1}{5}$ more than it cost. The selling price of the other was $\frac{1}{5}$ less than it cost. Was the total selling price of the two pianos more or less than their combined cost? How much?

7. A clock strikes the hour on the hour. How many strokes will it strike between 12:30 A.M. and 12:30 P.M.?

8. How can you determine the answer to Exercise 6 without listing and adding the numbers?

   HINT: The sequence of numbers is: 1,2,3, . . . 10,11,12
   Add the pairs: $1 + 12$, $2 + 11$, $3 + 10$, etc.
   What is the sum of each pair?
   Will the sum be the same for the remaining pairs?
   How many pairs will there be?
   What is the sum of all the numbers?

9. A *sequence* with the same difference, d, between consecutive terms is called an *arithmetic progression.* The first term is called a, and the last term l. The number of terms is n. In Exercise 7, what is a? What is l? What is n? What is d?

   Express the sum of $1 + 12$ in terms of a and l. 2 is $a + 1$. 11 is $l - 1$. What is the sum of this pair?
   Express the sum of $3 + 10$ in terms of a and l.
   How many pairs are there in terms of n?
   Express the sum of all the numbers in terms of a, l, and n.

10. Use the formula you developed in Exercise 8 to find the sum of the natural numbers from 1 to 40, inclusive.

11. What is the sum of the even numbers from 2 to 20, inclusive?

12. Write the terms of the arithmetic progression in which $n = 6$, $l = 25$, $a = 5$, and $d = 4$.

13. Write the terms of the arithmetic progression in which $a = 2$, $d = 3$, and $n = 7$.

14. You can find any given term in an arithmetic progression if you know the first term, a, and the common difference, d. The second term is $a + d$. The third term is $a + 2d$. What is the fourth term? the fifth?

15. You can see that to find any given term in an arithmetic progression we multiply d by one less than the number of the term, and add the product to a. Thus the nth term is $a + (n - 1)d$. What is the fifth term of an arithmetic progression in which $a = 2$ and $d = 5$? First calculate the answer using the formula. Then write the first five terms of the progression to see if it is correct.

16. The first three terms of an arithmetic progression are 6, 10, and 14.
    a. What is a? What is d?
    b. What is the next (fourth) term in the progression?
    c. What is the eighth term in the progression? First calculate the answer using the formula. Then check by listing the eight terms.

In some forms of cryptograms, or secret messages, letters may be replaced by numerals. To solve such a cryptogram it is necessary to restore the proper letters.

Here are some multiplication exercises in which the numerals are replaced by letters. By finding the proper clues, you can restore the numerals, and see what the exercise was.

The letter O is not used. When you see 0 it is zero.

EXAMPLE

```
   5 A B
     C 3
  ------
 D E E 6
F G G B
--------
F E B 9 6
```

Clue #1: $3 \times B = 6$. Then $B = 2$.
Clue #2: Since $3 \times 5 = 15$, then $D = 1$.
Clue #3: $E + B = 9$. Since $B = 2$, $E = 7$.
Clue #4: Then the first partial product is 1776.

The multiplicand must be 592.
Complete the solution.

You will find these easier than they look. First find the clues.

1.
```
  A 0 B
    C D
  -----
  D A D
C 2 4
-------
C 7 D D
```

2.
```
  E E 0
    7 8
  -----
  B B 0
A A 0
-------
B L B 0
```

3.
```
  Q 1 3
    4 6
-------
1 Q W S
S 5 Q
-------
T W T S
```

4.
```
    E A K
      L M
  -------
  K K S K
A L S 0
---------
1 7 8 3 5
```

5.
```
  C E D
    G J
  -----
  D D 1
2 5 7
-------
J J 4 G
```

6.
```
  B 0 Y
    2 5
-------
A R B 0
6 1 2
-------
G Y R 0
```

7.
```
  A E C
    E 2
  -----
  8 D 8
A E C
-------
5 P N 8
```

8.
```
    3 2 3
      A B
    -----
    C D C
1 6 1 5
---------
E F E E C
```

SPECIAL PROJECTS

1. Complete a multiplication exercise, and see how many digits you can replace with letters, still leaving enough clues so someone else can restore the digits.
2. Try the same thing with a division exercise.
3. Try the same thing with an addition exercise.
4. Try the same thing with a subtraction exercise.
5. In the Exercises above each of the digits is represented by its special letter. In Exercises 1, 3, and 5, try replacing all missing digits by *X*. How many of them can you solve?

Part One

A. Express each of the following as a per cent.

1. 0.27	**3.** 0.256	**5.** 1.6	**7.** 0.03	**9.** 0.005
2. 0.35	**4.** 0.875	**6.** 2.4	**8.** 1.25	**10.** 0.366

B. Express each of the following as a decimal.

1. 46%	**3.** 4%	**5.** 1.3%	**7.** 180%	**9.** 16.3%
2. 87%	**4.** 125%	**6.** 0.3%	**8.** 45.2%	**10.** 200%

C. Express each of the following as a per cent to the nearest tenth of 1%.

1. $\frac{1}{2}$	**4.** $\frac{4}{5}$	**7.** $\frac{1}{6}$	**10.** $\frac{1}{8}$	**13.** $\frac{1}{3}$
2. $\frac{7}{8}$	**5.** $\frac{1}{4}$	**8.** $\frac{5}{8}$	**11.** $\frac{5}{6}$	**14.** $\frac{5}{4}$
3. $\frac{2}{3}$	**6.** $\frac{3}{8}$	**9.** $\frac{3}{4}$	**12.** $\frac{7}{5}$	**15.** $\frac{3}{5}$

D. Find the value of N in each of the following:

1. 6 is $N\%$ of 15
2. $N\%$ of 45 is 18
3. 18 is $N\%$ of 20
4. $N\%$ of 36 is 27
5. $N\%$ of 48 is 6
6. 25 is $N\%$ of 40
7. 35 is $N\%$ of 50
8. $N\%$ of 32 is 40
9. 24 is $N\%$ of 16
10. 40 is $N\%$ of 15

E. Solve each of the following proportions for N.

1. $\frac{5}{3} = \frac{N}{9}$	**5.** $\frac{N}{6} = \frac{18}{36}$	**9.** $\frac{15}{N} = \frac{45}{54}$
2. $\frac{7}{N} = \frac{21}{36}$	**6.** $\frac{3}{N} = \frac{18}{24}$	**10.** $\frac{4}{9} = \frac{N}{36}$
3. $\frac{12}{8} = \frac{N}{4}$	**7.** $\frac{15}{N} = \frac{3}{2}$	**11.** $\frac{N}{8} = \frac{16}{12}$
4. $\frac{24}{N} = \frac{8}{3}$	**8.** $\frac{9}{7} = \frac{27}{N}$	**12.** $\frac{26}{12} = \frac{N}{6}$

F. Set up the proportion, and solve each of these problems.

1. How far can a plane fly in 5 hours if it is traveling at the rate of 1050 miles in 2 hours?

2. What will a dozen cans of tomato juice cost if 3 cans cost 73¢?

3. At the rate of 75 miles on 5 gallons of gasoline, how far will a car travel on 12 gallons?
4. The Census Bureau reports that the population of the country is increasing at the rate of 5 persons per minute. How many is this per 24-hour day?

Part Two

A. List the numerals 1 through 6 on a sheet of paper. Referring to the proportion on the right, below, write the answer to each of the questions after the corresponding numeral on your paper.

$$\frac{5}{8} = \frac{25}{40}$$

1. What are the means of the proportion?
2. What are the extremes?
3. What are the numerators of the fractions?
4. What are the denominators?
5. What is the product of the means?
6. What is the product of the extremes?
7. How do you know that the proportion is a true statement?

B. Express the ratio between each of the following pairs in three ways: as a fraction, as a decimal, and as a per cent.

1. 9 to 4
2. 5 to 20
3. 6 to 10
4. 160 to 50
5. 16 to 25
6. 15 to 9

C. For each pair in B above, state which number named is the denominator before the fraction is written in simplest form.

D. Express each of the following in three ways, as in B above.

1. 1 pint to a gallon
2. a dime to a dollar
3. 9 inches to 2 yards
4. 2 feet to 3 inches
5. 12 ounces to 3 lb.
6. 1 second to 1 hour
7. 1 gallon to 3 pints
8. 3 miles to 40 rods
9. \$1 to 75¢
10. 6 quarts to 2 bushels

Part Three

1. In Miss Henderson's class of 25 pupils, 12 are boys. What per cent of the class are boys?
2. Mike had 12 problems correct in a test of 15 problems. What per cent of the problems were correct?

3. Martha bought a pair of gloves, regularly priced at $5, for $4.50. What per cent of the regular price did she save?

4. The population of Glendale is 25,000. The population of Centralia is 30,000. Centralia is how many times as large as Glendale? (Express the ratio in three ways.)

5. Mike can run 80 yards in 10 seconds. If he could run 100 yards at the same rate, how long would he take?

6. The basketball team has won 16 games and lost 4. What per cent of the number of games played has it lost?

7. Last June, eggs were selling for 50¢ a dozen. They are now selling at 65¢ a dozen. The increase is what per cent of the price of last June?

8. Jane sells subscriptions to a magazine for $4. She receives 25¢ for each subscription. This is what per cent of the subscription price?

9. There are 15,000 registered voters in Oakdale. At a recent election 9,000 votes were cast. What per cent of the number of registered voters failed to vote?

10. Mary had 22 out of 25 words right in a spelling test, and 17 out of 20 problems right in an arithmetic test. What per cent did she have correct in each test? In which test did she have the greater per cent correct?

11. The population of Harrisville is 30,000. The population of Hudson is 75,000. The population of Harrisville is what per cent of the population of Hudson?

12. Jane has an allowance of $6 a week. She spent $3 on lunches, $1.50 on bus fare, and 75¢ for school supplies. What per cent did she spend for each purpose?

13. Baseball standings, batting averages, and many other athletic records are expressed in decimals because they make comparisons easier. The games won and lost by the three top teams in the city last year were:

	Games Won	*Games Lost*	*Ratio of Games Won* *Fraction*	*Decimal*
Johnson High School	7	2	$\frac{7}{9}$	.778
Washington High School	6	2	$\frac{6}{8}$	.750
Mechanic Arts High School	7	3	$\frac{7}{10}$	.700

Fractions are not ordinarily used in such tables. Can you see why? Which team had the best record?

Part One

A. *Add:*

1. 28	**2.** 435	**3.** $2.27	**4.** $29.15
37	622	4.98	67.32
52	591	3.49	56.87
66	386	.74	43.26
41	714	8.44	9.81
34	887	5.25	136.17
19	645	6.66	75.56

B. *Subtract:*

1. 6324	**3.** 392,651	**5.** 786.29 ft.
2782	84,786	567.37 ft.
2. $300.00	**4.** $28.73	**6.** $522.33
212.98	19.76	348.92

Part Two

A. *Multiply:*

1. 4961×83 **3.** 8686×393 **5.** 74.26×5.27

2. 6723×415 **4.** $22.14 \times .03$ **6.** $\$12.98 \times 3.98$

B. *Divide.* Round to the nearest tenth where necessary.

1. $85 \div 6$ **3.** $2781 \div 9$ **5.** $82.4 \div 4.12$

2. $96 \div 8$ **4.** $5634 \div 14$ **6.** $783.64 \div 2.6$

Part Three

1. Find the products with as little written work as possible.

a. $200 \times .07$ **c.** $.004 \times 38$ **e.** $200 \times \$4.20$

b. $500 \times .025$ **d.** $.06 \times 25$ **f.** 3000×19

2. Find the quotients with as little written work as possible.

a. $27 \div .03$ **c.** $563 \div .01$ **e.** $\$84.00 \div .07$

b. $186 \div 100$ **d.** $4728 \div .004$ **f.** $\$51.00 \div 300$

Part Four

Copy the products and place the decimal point correctly. Insert zeros, as needed.

1. $2.63 \times 4.6 = 12098$

2. $26.3 \times 0.046 = 12098$

3. $.263 \times 0.046 = 12098$

4. $26.3 \times 0.46 = 12098$

5. $2.63 \times 0.0046 = 12098$

6. $263 \times 4.6 = 12098$

Part Five

A. *Add.* Write all answers in simplest form.

1. $\frac{1}{7} + \frac{3}{7}$

2. $\frac{2}{5} + \frac{3}{10}$

3. $4\frac{1}{8} + 3\frac{5}{8}$

4. $9\frac{1}{2} + 6\frac{1}{6}$

5. $12\frac{3}{8} + 9\frac{1}{3}$

6. $3\frac{1}{3} + 4\frac{2}{5} + 8\frac{1}{7}$

7. $2\frac{2}{3} + 5\frac{1}{2} + 3\frac{5}{6}$

8. $3\frac{1}{4} + 7\frac{3}{8} + 4\frac{7}{16}$

9. $2\frac{5}{12} + 6\frac{3}{4} + 5\frac{2}{3}$

B. *Subtract.* Write all answers in simplest form.

1. $\frac{7}{9} - \frac{5}{9}$

2. $12\frac{8}{13} - 9\frac{9}{13}$

3. $10 - 6\frac{5}{6}$

4. $8\frac{5}{8} - 3\frac{5}{6}$

5. $22\frac{1}{2} - 14\frac{1}{7}$

6. $8\frac{3}{13} - 4\frac{2}{9}$

Part Six

A. *Multiply.* Write all answers in simplest form.

1. $\frac{1}{3} \times \frac{3}{8}$

2. $\frac{1}{4} \times \frac{16}{21}$

3. $\frac{2}{3} \times \frac{9}{14}$

4. $15 \times \frac{4}{5}$

5. $14 \times 3\frac{3}{7}$

6. $12\frac{1}{2} \times 4\frac{2}{25}$

B. *Divide.* Write all answers in simplest form.

1. $\frac{2}{3} \div \frac{1}{6}$

2. $6\frac{1}{4} \div 12\frac{1}{2}$

3. $2\frac{3}{5} \div 10$

4. $3\frac{3}{5} \div 16\frac{2}{3}$

5. $\frac{7}{15} \div 2\frac{1}{10}$

6. $12\frac{1}{2} \div 3\frac{4}{7}$

Part Seven

1. Express as decimals.

a. 26%
b. 7.8%
c. 62.5%
d. 33.67%
e. 185%
f. 0.02%

2. Express as per cents.

a. 0.12
b. 2.65
c. 0.512
d. 4.04
e. 0.07
f. 3.62

Part Eight

Express the ratio of each of the following pairs in three ways: as a fraction in simplest form; as a decimal rounded to the nearest thousandth; and as a per cent to the nearest tenth of 1%.

1. 3 to 12
2. 5 to 6
3. 8 to 5
4. 42 to 48
5. 28 to 14
6. 60 to 45

Part Nine

Find the value of the variable, N, in each of the following:

1. 3 is N% of 5
2. N% of 24 is 18
3. 16 is N% of 72
4. 25 is N% of 125
5. 21 is N% of 30
6. N% of 64 is 40

Part Ten

In league games during the basketball season the High Point High School team scored the following number of points per game: 46, 26, 54, 36, 62, 75, 82. Draw a line graph to show these scores.

Part Eleven

1. Margaret works $27\frac{1}{2}$ hours per week during the summer, and is paid \$1.10 per hour. She saves 0.7 of her earnings. How much per week does she save?

2. Last year Fred gained $5\frac{13}{16}$ pounds. This year he gained $8\frac{1}{8}$ pounds. How much more did he gain this year than last year?

3. The three sides of a triangle measure $3\frac{1}{8}$ inches, $4\frac{3}{4}$ inches, and $2\frac{15}{16}$ inches respectively. What is the perimeter of the triangle?

4. A regular pentagon has sides each measuring $2\frac{3}{8}$ inches long. What is its perimeter?

5. Alice bought a party dress at a clearance sale for \$11. The regular price of the dress was \$18.50. What per cent of the regular price of the dress did she save?

6. Last year Mr. Willis earned \$7200. This year he earned \$7800. His increase in earnings is what per cent of last year's earnings?

7. A salesman received a commission of \$36.25 on a sale of \$725 worth of goods. At that rate what would be his commission on a sale of \$2450?

HOW WE USE ALGEBRA

WORDS TO WATCH FOR

absolute value
additive inverse
associative property
coefficient
commutative property
constant
distributive property
equation
formula
integer
multiplicative inverse
negative integer
negative number
parentheses
positive integer
positive number
rational number
reciprocal
signed number
variable

At 7:00 P.M. the temperature at a town in northern Minnesota was 20° above zero. By 6:00 A.M. it had dropped 16°. What was the temperature then? The conditional statement is:

$$20 - 16 = n \qquad\qquad s - a = b$$

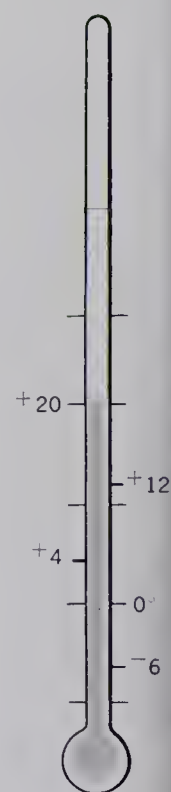
Then $\quad n = 4$

We can check the answer by adding $a + b$: $16 + 4 = 20$.

The next night a cold wind from Canada lowered the temperature. At 7:00 P.M. the temperature was 12° above zero. By 6:00 A.M. it had dropped 18°. What was the temperature at 6:00 A.M.? The conditional statement is:

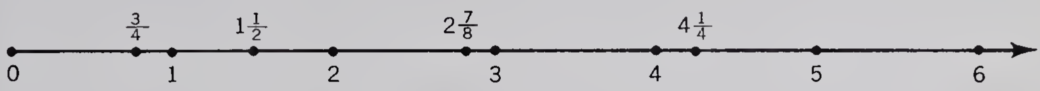
$$12 - 18 = n \qquad\qquad s - a = b$$

You can see, by comparing this statement to the previous statement, that n and 18 are addends, and 12 is the sum. However, notice that while you can readily find the answer

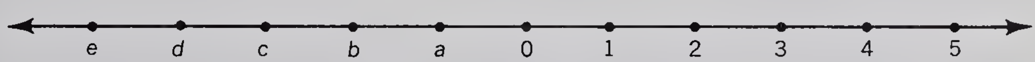


on the thermometer (the Figure on page 248), there is no number in the set we have been using (the set of fractional numbers of arithmetic, which of course includes the whole numbers as a subset) that will satisfy the conditions of the statement.

It is evident that to satisfy the conditions of such a statement, we need to extend the number system and the number line that represents it. We are familiar with the number line in representing the fractional numbers of arithmetic (zero and numbers greater than zero). Any number represented on the number line is greater than any number to its left and less than any number to its right.



The point associated with 6, for example, is to the right of $4\frac{1}{4}$. Then $6 > 4\frac{1}{4}$. In like manner $\frac{2}{3} > 0$. We can use the same relationships to extend the number line to the left. Then zero, instead of being the origin, is the point of reference. What can you say about the value of all points to the left of zero, as compared to the value of zero?



The point a on the number line above is the same distance to the left of 0 as the point associated with 1 is to the right of 0. The distance of a number from 0 on the number line determines the *absolute value* of the number. This is its numerical value disregarding its direction from 0. Since a is the same distance from 0 as is 1, we can denote its absolute value as $|\,1\,|$. The two lines indicate that we are concerned only with distance from 0 and are disregarding direction from 0.

Since a is to the left of 0, its value must be less than 0. Thus $a < 0 < 1$. To provide for this difference in value, numerals to the left of 0 represent *negative numbers,* and are written with a symbol similar to the minus sign. Thus the number associated with a is $^-1$. Note that the minus symbol is "raised." Numerals to the right of 0 represent *positive numbers* and may be written with a raised plus symbol, as $^+1$. A numeral written with no sign represents a positive number.

Therefore, for each fractional number of arithmetic there is a "mate" in the negative numbers. That is, for each positive number there is a negative number that has the same absolute value. For example, the absolute value of $^-1$ is 1, just as the absolute value of 1 is 1. In other words

$$|\,^-1\,| = 1 \qquad \text{and} \qquad |\,1\,| = 1$$

Therefore $$|\,^-1\,| = |\,1\,|$$

With this extension of the number line, we can return to the solution of the conditional statement:

$$12 - 18 = n$$

Note that, on the number line, to subtract one number from another you move the indicated number of units to the left. Moving 18 units to the left of 12 you find $^{-}6$. Then $n = {}^{-}6$.

1. Numerals written with + and – symbols represent *signed numbers.* What signed numbers are associated with b and c in the Figure on page 249?

Set up the conditional statement for each of these problems and solve.

2. It was $^{-}5°$ at 4:00 A.M. In the next seven hours the temperature rose 17°. What was the temperature at 11:00 A.M.?

3. A sailboat sailed south for 6 miles, then reversed its course and sailed 27 miles to the north. How many miles was it to the north of its starting point?

4. Mike withdrew \$17 from his savings account on June 1, then deposited \$25 on June 15. How much was his balance increased during the month?

5. The Colts have the ball on the 20-yard line in a football game, with 10 yards to go for the first down. On the first play they lose 3 yards; on the second play they gain 10 yards. How many yards are left now for a first down?

6. The temperature at midnight was 11 degrees. By 3:00 A.M. it had dropped 15 degrees. What was the temperature at 3:00 A.M.?

7. The temperature at Salina was 38 degrees while at Boise it was 5 degrees below zero. How much higher was the temperature at Salina?

8. Harry had \$7.50. He paid a debt of \$3.75, and Fred paid Harry back \$1.25 which he borrowed. What did Harry then have?

9. An airplane traveled 400 miles due north, then 550 miles due east, then 400 miles due south, and finally 350 miles due west. How far was the airplane from its starting point?

Copy each pair below, and insert the symbol < or > between each pair in order to make a true statement:

10. $^{+}2$, $^{+}\frac{1}{2}$

11. $^{+}2$, $^{+}5$

12. $^{-}5$, $^{-}1$

13. $^{-}113$, $^{+}3$

14. $^{-}3\frac{1}{2}$, $^{-}3$

15. $^{-}6$, $^{-}8$

16. $^{+}3$, $^{+}4$

17. $^{-}3$, $^{+}3$

18. $^{+}18$, $^{+}23$

19. $^{-}\frac{7}{3}$, $^{-}\frac{1}{4}$

20. $^{+}\frac{5}{8}$, $^{+}\frac{1}{2}$

21. $^{+}\frac{7}{6}$, $^{-}2$

22. $^{+}2$, 0

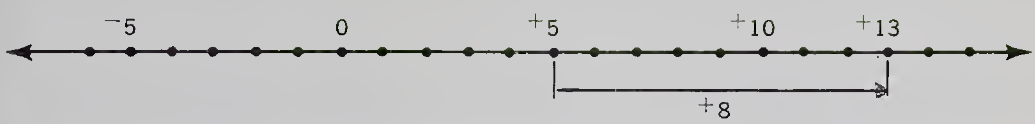
23. $^{-}6$, $^{-}3$

24. $^{+}7$, $^{-}2$

In operations with signed numbers you are concerned not only with absolute value but also with "direction." The direction of the number is indicated by its sign. A positive number is directed to the *right* of zero. You are accustomed to this, although you may not have thought of it in this way.

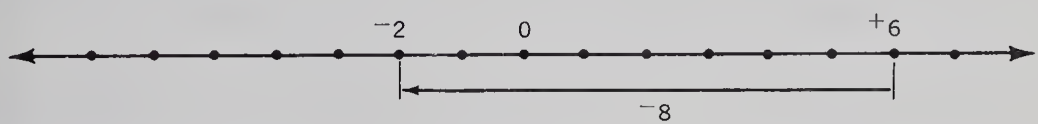
EXAMPLES

1. Add: $^{+}5 + (^{+}8)$



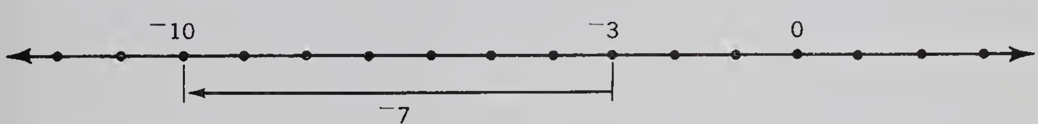
The length of the segment representing $^{+}8$ indicates its absolute value, $|8|$. (Its direction is to the right since it is positive.) Hence, as you expect, the sum is $^{+}13$.

2. Add: $^{+}6 + (^{-}8)$



Explain the value and direction of $^{+}6$ and $^{-}8$. What is the sum?

3. Add: $^{-}3 + (^{-}7)$



Explain the value and direction of $^{-}3$ and $^{-}7$. What is the sum?

1. Sketch a number line as in the Examples above to perform the following additions.

a. $6 + (^{+}3)$
b. $8 + (^{-}9)$
c. $^{-}2 + (^{-}6)$
d. $^{-}4 + (^{-}9)$
e. $3 + (^{+}2)$
f. $9 + (^{-}12)$
g. $7 + (^{+}5)$
h. $^{-}3 + (^{-}7)$
i. $10 + (^{-}6)$

2. Find as many examples as you can from Exercise 1 to illustrate this Rule:

Rule: To add two signed numbers with like signs, add their absolute values and give the result the common sign.

3. Find as many examples as you can in Exercise 1 to illustrate this Rule:

Rule: To add two numbers with unlike signs, subtract the lesser absolute value from the greater, and give the result the sign of the number with the greater absolute value.

---

4. Use the Rules as stated in Exercises 2 and 3 in performing the following additions:

EXAMPLES

1. $$\begin{aligned} {}^{+}6 + ({}^{+}4) &= |{}^{+}6| + |{}^{+}4| \\ &= {}^{+}(6 + 4) \\ &= {}^{+}(10) = {}^{+}10 \end{aligned}$$

2. $$\begin{aligned} {}^{-}5 + ({}^{-}2) &= |{}^{-}5| + |{}^{-}2| \\ &= {}^{-}(5 + 2) \\ &= {}^{-}(7) = {}^{-}7 \end{aligned}$$

3. $$\begin{aligned} {}^{+}4 + ({}^{-}8) &= |{}^{-}8| - |{}^{+}4| \\ &= {}^{-}(8 - 4) \\ &= {}^{-}(4) = {}^{-}4 \end{aligned}$$

4. $$\begin{aligned} {}^{+}7 + ({}^{-}3) &= |{}^{+}7| - |{}^{-}3| \\ &= {}^{+}(7 - 3) \\ &= {}^{+}(4) = {}^{+}4 \end{aligned}$$

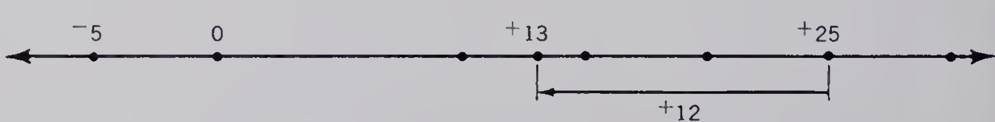
---

a. ${}^{-}18 + ({}^{-}23)$
b. $14 + ({}^{-}20)$
c. ${}^{-}16 + ({}^{-}24)$
d. ${}^{-}17 + ({}^{+}10)$
e. $13 + ({}^{+}27)$
f. $32 + ({}^{-}19)$
g. $28 + ({}^{+}27)$
h. ${}^{-}22 + ({}^{-}15)$
i. ${}^{-}45 + ({}^{-}15)$
j. ${}^{-}45 + ({}^{-}16)$
k. $16 + ({}^{-}21)$
l. ${}^{-}71 + ({}^{+}47)$
m. ${}^{-}11 + ({}^{+}11)$
n. ${}^{-}101 + ({}^{-}24)$
o. ${}^{+}44 + ({}^{-}23)$
p. ${}^{+}\frac{1}{2} + ({}^{-}\frac{1}{4})$
q. ${}^{+}38 + ({}^{+}72)$
r. ${}^{-}76 + ({}^{+}76)$

5. In a subtraction exercise you are given a sum and a known addend, and you are to find the unknown addend. In the statement:

$${}^{+}25 - ({}^{+}13) = N \qquad\qquad s - a = b$$

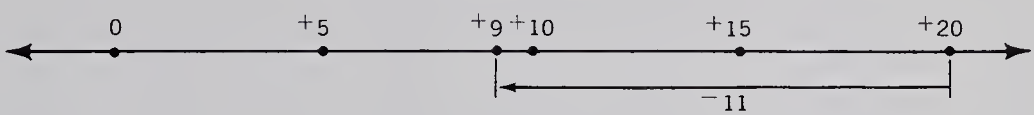
${}^{+}25$ is the sum and ${}^{+}13$ is the known addend. The question is, What number added to ${}^{+}13$ gives ${}^{+}25$ as the sum or ${}^{+}13 + N = {}^{+}25$? Referring to the number line, you see that the absolute value of the unknown addend is 12. Since the sum is to the right of the known addend, the sign of the unknown addend is +.



EXAMPLES

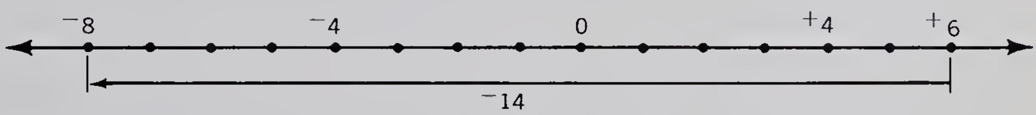
1. Find: $^+9 - (^+20) = N$

   $^+9$ is the sum, and $^+20$ is the given addend, therefore, $^+20 + N = {}^+9$. Referring to the number line you see that the absolute value of the unknown addend is 11. The sum is in what direction from the known addend? What is the sign of the unknown addend?



2. Find: $^-8 - (^+6) = N$

   Refer to the number line. What is the absolute value of the unknown addend? What is its sign?



Sketch a number line for reference in performing the following subtractions.

a. $15 - (^-6)$ **c.** $16 - (^-5)$ **e.** $9 - (^+25)$ **g.** $7 - (^+15)$
b. $^-5 - (^-11)$ **d.** $^-3 - (^+8)$ **f.** $^-8 - (^+15)$ **h.** $^-5 - (^+18)$

6. It is convenient to replace the operation of subtraction by an equivalent addition by applying this Rule:

Rule: Add to the minuend the number "mate" of the subtrahend using the rules for adding signed numbers.

7. Use the Rule to perform the following subtractions.

EXAMPLES

1. $^+18 - (^+6) = {}^+18 + (^-6) = {}^+12$
2. $^-13 - (^+15) = {}^-13 + (^-15) = {}^-28$
3. $^+9 - (^+12) = {}^+9 + (^-12) = {}^-3$
4. $^-7 - (^-6) = {}^-7 + (^+6) = {}^-1$

a. $18 - (^-21)$ **c.** $32 - (^-19)$ **e.** $45 - (^+26)$ **g.** $26 - (^-19)$
b. $^-15 - (^+16)$ **d.** $^-31 - (^-31)$ **f.** $^-17 - (^+22)$ **h.** $^-29 - (^-31)$

Signed numbers are very convenient for expressing values in business and other activities where it is desirable to indicate direction as well as amount. For example, in the daily account of sales of shares on the stock exchange an increase in price over the previous day is indicated by +, and a decrease by −, in the column headed N Ch (Net Change).

Sales	*Stocks*	*Close*	*N Ch.*	*Sales*	*Stocks*	*Close*	*N Ch.*
270,600	Am Photo	$12\frac{1}{4}$	$^{+}1\frac{1}{8}$	61,100	Cone Mills	$15\frac{3}{4}$	$^{+}1\frac{3}{8}$
113,700	R C A	$95\frac{1}{4}$	$^{+}3\frac{3}{8}$	44,100	Martin M	$19\frac{1}{2}$	$^{+}\frac{3}{4}$
106,000	Gen Motr	$87\frac{3}{4}$	$^{+}\frac{3}{8}$	40,900	Raytheon	$25\frac{1}{8}$	$^{-}\frac{1}{2}$
82,000	Sunra DX	34	$^{-}2\frac{1}{2}$	40,800	Zenith	$81\frac{3}{8}$	$^{+}3\frac{5}{8}$
79,700	Pen Ford	$24\frac{1}{4}$	$^{+}\frac{5}{8}$	40,700	Ford Motr	$52\frac{1}{8}$	$^{+}\frac{1}{8}$
71,700	Sperry R	$17\frac{1}{4}$	unch	40,200	Yale Town	33	$^{-}\frac{1}{4}$
64,400	Chrysler	$93\frac{1}{4}$	$^{+}1\frac{1}{4}$	39,300	Gulf Oil	$47\frac{1}{8}$	$^{-}\frac{3}{8}$

1. In the quotation for RCA in the illustration above, what is meant by $^{+}3\frac{3}{8}$?

2. The column headed "Close" gives the day's final price for one share of stock. Thus, $12\frac{1}{4}$ means that \$12.25 was paid for one share of stock at the "close" of the day. What was the closing price of RCA stock on the previous day?

3. Suppose the net change had been $^{-}2$ instead of $^{+}3\frac{3}{8}$, what would have been the closing price listed?

4. Which stocks closed lower than they did on the previous day?

5. Which stock made the greatest gain during the day?

6. Signed numbers are also convenient for representing deposits and withdrawals from a checking or savings account. If a deposit of \$10 is represented by $^{+}10$, how would a withdrawal of \$25 be represented?

7. How do you represent 15° below zero with a signed number?

8. If it is $^{-}12°$ at 6:00 A.M. and the temperature rises 20° by noon, use a signed number to represent the temperature at noon.

9. If the temperature at New Orleans is 68° and the temperature at Minneapolis is $^{-}13°$, how many degrees warmer is it at New Orleans than at Minneapolis?

10. If the temperature is 8° above zero at 6:00 P.M. and falls 20° in the next twelve hours, write the 6:00 A.M. temperature as a signed number.

11. If the temperature at Hillsdale was 3 degrees below zero at 4:00 A.M. but rose 32 degrees by 3:00 P.M., what was the temperature at 3:00 P.M.?

You have learned that a statement like

$$11 + 16 > 25$$

is a mathematical sentence. Every sentence has a verb. The verbs most commonly used in mathematical sentences are:

$>$, is greater than
$<$, is less than
$=$, is equal to
$\neq$, is not equal to

You will recall that a mathematical sentence may be true, false, or conditional. Write each of these mathematical sentences in words, and then state whether it is true, false, or conditional.

1. $13 + 5 = 18$
2. $15 + 6 \neq 8 + 13$
3. $25 + 16 > 40$
4. $16 + N < 20 + 2$
5. $9 + 3 > 5 + 6$
6. $19 = 3 + 16$
7. $23 = 17 + N$
8. $18 + 4 < 9 + 15$
9. $21 + 3 \neq 14 + N$
10. $16 + 7 > 5 + 18$
11. $18 + 6 = 15 + 9$
12. $5 + 9 \neq 8 + 7$
13. $21 + 3 = 42 - N$
14. $12 + 15 > 18 + 9$
15. $4 + 3 < 6 + 2$
16. $17 - 9 = 5 + 3$
17. $13 + 11 \neq 16 + 9$
18. $5 + N = 3 \times 3$

19. Using each of the verbs, write a true mathematical sentence, in symbols.
20. Write four false mathematical sentences, in symbols, one with each of the verbs.
21. Use each of the verbs in writing a conditional sentence in symbols.

Translate each of the following sentences into symbols. State whether each is true, false, or conditional.

22. The sum of 5 and 18 is greater than 20.
23. If a certain number is divided by 7 the quotient is 9.
24. One-fifth of a certain number is greater than 15.
25. The ratio of 5 to 7 is equal to the ratio of 15 to 21.
26. The sum of 15 and 19 is less than 30.
27. If a certain number is added to 20 the sum is equal to the product of 5 and 7.
28. If 27 is divided by 3 the quotient is not equal to 8.
29. Forty-eight is the product of 3 and 17.
30. The sum of 9 and 16 is greater than the product of 5 and 7.

A mathematical sentence in which the verb is = is an *equation*. An equation may be true, false, or conditional. When it is true, both sides of the equation name the same number. When it is conditional, a number must be found that can be substituted for the variable so as to make the equation true. You have learned how to do this in solving conditional equations describing the operations of addition and multiplication.

1. In this equation the unknown is an addend:

$$N + 15 = 37 \qquad\qquad a + b = s$$

Write the equivalent equation in which the variable is alone on one side. Then find the value for N that makes the equation true.

2. In this equation a factor is unknown:

$$56 \div N = 8 \qquad\qquad p \div x = y$$

Write an equivalent equation in which the unknown is alone on one side. Then find the value for the unknown that makes the equation true.

For each of the following write the equivalent equation that has the variable alone on one side. Then find the value for N that makes the equation true.

3. $N \div 3 = 7$

4. $4N = 28$

5. $\frac{16}{N} = 2$

6. $18 - N = 5$

7. $\frac{5}{N} = \frac{15}{27}$

8. $42 = 7N$

9. $N - 8 = 9$

10. $12 = \frac{48}{N}$

11. $\frac{3}{5} = \frac{9}{N}$

12. $26 = N + 3$

13. $3 = 17 - N$

14. $36 = 9N$

15. $\frac{56}{\mathrm{N}} = 7$

16. $16 = 9 + N$

17. $8N = 72$

18. $27 \div N = 9$

19. $25 - N = 4$

20. $\frac{N}{4} = \frac{20}{16}$

21. $N - 16 = 3$

22. $N - 11 = 16$

Translate each of the following statements into an equation, then solve the equation.

23. The sum of two numbers is 52. One of the numbers is 18. What is the other number?

24. The product of two factors is 57. One of the factors is 3. What is the other factor?

25. If a certain number is subtracted from 19, the result is 4. What is the number?

26. The ratio of 9 to 5 is the same as the ratio of a certain number to 15. What is the number?

27. A sweater is on sale for $12, which is $\frac{4}{5}$ of the regular price. What is the regular price?

28. Jim spent $\frac{1}{6}$ of his savings for a new suit which cost $55. How much were his savings before he bought the suit?

29. The sum of 19 and 17 is the same as twice a certain number. What is the number?

30. If a certain number is divided by 7, the result is equal to the sum of 18 and 6. What is the number?

31. Five times a certain number is 8 less than 53. What is the number?

32. Three times a number is 27. What is the number?

33. One-fourth of a number is 50. What is the number?

34. A number decreased by 15 gives 65. Find the number.

35. A number increased by 7 gives 21. What is the number?

36. Seven more than a certain number is 56. Find the number.

37. A bicycle was on sale for $50.00 which was $\frac{2}{3}$ the regular price. What was the regular price of the bicycle?

38. Six times a number decreased by 11 equals 7. Find the number.

39. If a certain number is divided by 5, the quotient is equal to the difference between 11 and 3. Find the number.

40. A number increased by 14 gives 45. What is the number?

41. Twice a certain number decreased by 18 gives a result equal to half of 44. Find the number.

42. Half a certain number increased by 7 equals 20. Find the number.

43. If a number is decreased by 4, the result is five more than 22. What is the number?

44. Twice a certain number is decreased by 16 to give a result of 48. What is the number?

45. Seven times a number is 35. What is the number?

46. If a number is doubled and then decreased by 7, the result is 19. What is the number?

47. One-third of a certain number is 14. Find the number.

48. Eleven more than a number is 40. Find the number.

49. Mary spent $\frac{1}{4}$ of her summer earnings on new clothes. If she spent $85 on new clothes, what were her earnings?

50. The sum of 58 and 32 is three times a certain number. What is the number?

51. A number increased by 6 gives a result of 40. Find the number.

Additive and Multiplicative Inverses

1. Perform the following additions and subtractions.

a. $6 + (^-3)$	**f.** $11 - (^+3)$	**k.** $^-17 + (^+6)$
b. $8 - (^+2)$	**g.** $^-13 + (^+13)$	**l.** $^-12 - (^+3)$
c. $9 + (^+3)$	**h.** $8 - (^+10)$	**m.** $14 + (^-14)$
d. $16 - (^-7)$	**i.** $17 - (^-3)$	**n.** $^-12 - (^-14)$
e. $6 + (^-6)$	**j.** $5 + (^-9)$	**o.** $^-3 + (^-9)$

2. Here are some generalizations about the operations you have been performing. See how many examples you can find in Exercise 1 above that illustrate each generalization. Let a represent any positive or negative number or zero.

a. $+(^-a) = {}^-a$	**c.** $+(^+a) = {}^+a$	**e.** $^+a + (^-a) = 0$
b. $-(^+a) = {}^-a$	**d.** $-(^-a) = {}^+a$	**f.** $^-a + (^+a) = 0$

3. If the sum of any two numbers is zero, then the numbers are *additive inverses* of each other. See e, g, and m in Exercise 1. Write the additive inverse for each of these numbers:

a. 15	**e.** $^-7$	**i.** ^-y
b. 12	**f.** n	**j.** 16
c. 26	**g.** ^-b	**k.** ^-n
d. $^-9$	**h.** $^-19$	**l.** x

4. Sketch a number line to illustrate how the distance from zero to a number compares with that of its additive inverse. What can you say about the absolute values of a number and its additive inverse?

5. How does the direction of a number from zero compare with that of its additive inverse?

6. The Rule for subtraction of signed numbers may be stated as follows:

> To subtract signed numbers, add the additive inverse of the subtrahend to the minuend.

Find five illustrations of this Rule in Exercise 1 above.

7. Do you recall what is meant by the *reciprocal* of a number? The reciprocal is also called the *multiplicative inverse* of a number. (See page 132 in Chapter 4.) Therefore, the product of a number and its multiplicative inverse is 1, except when the number is zero. Why do we exclude zero?

EXAMPLES

1. Find the multiplicative inverse of 23.

If N is the multiplicative inverse, $23N = 1$.
1 is the product, and 23 is the known factor. Then,

$$N = 1 \div 23, \qquad \text{or} \qquad N = \frac{1}{23}$$

2. Find the multiplicative inverse of $\frac{5}{8}$.

$\frac{5}{8}N = 1$ or $N = 1 \div \frac{5}{8}$ Then: $N = \frac{8}{5}$

3. Find the multiplicative inverse of 4.9.

$4.9N = 1$ Since $4.9 = 4\frac{9}{10} = \frac{49}{10}$,
then $N = 1 \div \frac{49}{10}$
and $N = \frac{10}{49}$

Find the multiplicative inverse of each of the following:

a. 7	**g.** $\frac{4}{5}$	**m.** 2.5
b. 13	**h.** $\frac{9}{10}$	**n.** 0.5
c. 22	**i.** $\frac{7}{12}$	**o.** 3.7
d. 6	**j.** $\frac{14}{15}$	**p.** 8.2
e. 3	**k.** $1\frac{2}{3}$	**q.** 7.8
f. $\frac{2}{3}$	**l.** $\frac{1}{6}$	**r.** 5.3

8. Following are several statements dealing with the multiplicative inverse. Answer whether the statements are true or false.

a. We can find the multiplicative inverse of every number.
b. Another name for reciprocal is multiplicative inverse.
c. The reciprocal of a number is always less than the number.
d. The reciprocal of $7\frac{1}{2}$ is $\frac{2}{15}$.
e. The multiplicative inverse of a number is never equal to the number.
f. The reciprocal of $\frac{17}{51}$ is twice the reciprocal of $\frac{2}{3}$.
g. The sum of the reciprocals of 3 and 4 is $\frac{1}{2}$.
h. The reciprocal of 5 is greater than the reciprocal of 2.
i. The product of a number and its reciprocal is 1.
j. The product of the reciprocals of 4 and 5 is less than the sum of their reciprocals.
k. The only number whose reciprocal is undefined is zero.
l. The difference between the multiplicative inverses of 2 and 4 is $\frac{1}{2}$.
m. Thirteen is greater than 10; therefore, its reciprocal is less than the reciprocal of 10.
n. The sum of the reciprocals of $\frac{1}{2}$ and $\frac{1}{4}$ increased by 5 gives a result less than 4.

The equations that you have learned to solve are those that express relationships between addends and sum, or between factors and product, or are statements of proportion. There are many equations that do not fit into the special patterns.

First let us examine an equation and review some special terms:

$$3x - 7 = x + 9$$

As you know, x is the *variable*. While many values may be assigned to the variable, only one of them will make this particular equation true. A number used as a multiplier of the variable is called the *coefficient* of the variable. On the left side of the equation the coefficient of x is 3. When the variable is written without a coefficient the coefficient is understood to be 1. So on the right side of the equation the coefficient of x is 1. The numbers 7 and 9 are called *constants*.

Both sides of a true equation name the same number. Both sides may be multiplied by the same factor, or increased or decreased by the same amount, and still name the same number. The procedure for solving an equation makes use of this property. We can use the equation written above as an illustration of this procedure.

First, we always want the variable alone on one side of the equation, and the constant alone on the other side. To achieve this, we can use the additive inverses to remove the constant from the left side and the variable from the right side. Remember always to perform the same operation on both sides.

(1) Rewrite the equation as: $3x + (^-7) = x + 9$

(2) Add the additive inverse of $(^-7)$ to both sides:

$$3x + (^-7 + 7) = x + (9 + 7)$$

Then: $$3x = x + 16$$

(3) Add the additive inverse of x, that is, (^-x), to both sides:

$$3x + (^-x) = (x + {^-x}) + 16$$

Then: $$2x = 16$$

(4) To find what x equals, we multiply both sides by the multiplicative inverse of the coefficient of x, that is, $\frac{1}{2}$:

$$\tfrac{1}{2} \times 2x = \tfrac{1}{2} \times 16$$

Then $$x = 8$$

When the coefficient of the variable is a proper or improper fraction, the variable is often written as part of the numerator. Thus, $\frac{3}{4}x$ may be written as $\frac{3x}{4}$; $\frac{8}{3}x$ as $\frac{8x}{3}$, etc. Do not let this confuse you in using

the multiplicative inverse. Until you get used to it you will find it helpful to rewrite the coefficient separately from the variable.

It is important while you are practicing this procedure to explain each step, as shown above, and in this Example.

EXAMPLE

Solve: $\frac{5x}{4} - 3 = \frac{2x}{3} + 4$

(1) Rewrite the equation: $\frac{5}{4}x + (^-3) = \frac{2}{3}x + 4$

(2) Use the additive inverse to remove the constant, $^-3$, from the left side:

$$\frac{5}{4}x + (^-3 + 3) = \frac{2}{3}x + (4 + 3)$$

Then: $$\frac{5}{4}x = \frac{2}{3}x + 7$$

(3) Use the additive inverse to remove the variable, $\frac{2}{3}x$, from the right side:

$$\frac{5}{4}x + (^-\tfrac{2}{3}x) = (\tfrac{2}{3}x + {}^-\tfrac{2}{3}x) + 7$$

Then: $$\frac{7}{12}x = 7$$

(4) Use the multiplicative inverse to make the coefficient of x equal to 1:

$$\frac{12}{7} \times \frac{7}{12}x = \frac{12}{7} \times 7$$

Then: $$x = 12$$

Write each step in solving the following equations.

1. $2x + 4 = 12$

2. $3x - 2 = 10$

3. $4n - 6 = 2$

4. $5x + 4 = 34$

5. $3 + 6x = 15$

6. $\frac{a}{2} + 3 = 4$

7. $2r - 5 = 1$

8. $4 + 4x = 16$

9. $x - 6 = 12 - x$

10. $\frac{2x}{3} + 4\frac{1}{3} = 11$

11. $3x = 12 - x$

12. $2\frac{1}{2}a + 6 = 9\frac{1}{2}$

13. $7x + 3 = 5x + 13$

14. $2x - 5 = x - 2$

15. $2x + 6 = \frac{x}{2} + 13.5$

16. $\frac{2x}{3} - 10 = \frac{x}{3} - 2$

17. $\frac{2x}{3} + 1\frac{1}{3} = x - 1$

18. $\frac{3x}{4} - \frac{5}{4} = \frac{x}{4} + \frac{1}{4}$

19. $\frac{5x}{2} - 3 = 2x$

20. $\frac{5x}{2} + 4 = 13 - \frac{x}{2}$

21. $\frac{3x}{4} + 9 = \frac{x}{2} + 15$

22. $4x - 10 = 2x - 2$

23. $\frac{x}{2} + 3 = 8$

Oral exercises: Expressing Ideas Algebraically

1. A square field has a perimeter of 28 ft. What is the length of one side?

a. If one side has a length s, how can you express the perimeter?
b. What is the equation? **c.** How long is a side?

2. The perimeter of an equilateral triangle is 18 rods. What is the measure of each side?

a. If one side of the triangle has a length s, express the perimeter.
b. What is the equation? **c.** How long is each side?

3. A rectangular field is twice as long as it is wide. The perimeter is 360 rods. What are the length and width?

a. If the width is n rods, express the length in terms of n.
b. Express the perimeter in terms of n.
c. What is the equation?
d. How many rods is n, which is the width?
e. How many rods is the length?

4. The length of a rectangular field is 30 rods more than the width. The perimeter is 180 rods. What are the length and the width?

a. If the width is n rods, express the length in terms of n.
b. Express the perimeter in terms of n.
c. What is the equation?
d. How many rods is n, which is the width?
e. How many rods is the length of the field?

5. The measures of the equal sides of an isosceles triangle are each twice the measure of the base. The perimeter of the triangle is 25 inches. How long is each of the sides?

a. If the base is x inches long, how long is each of the equal sides, in terms of x?
b. What is the perimeter of the triangle, in terms of x?
c. What is the equation?
d. How long is the base of the triangle?
e. How long is each of the equal sides of the triangle?

6. A regular octagon has a perimeter of 640″. How long is each side?

a. What is the perimeter if one side is s?
b. Write and solve the equation.

7. The length of each of the sides of equal measure of an isosceles triangle is 5 inches greater than that of the base. The perimeter of the triangle is 40 inches. What is the length of each side of the triangle?

a. If the base of the triangle is b inches long, how long is each of the equal sides in terms of b?
b. What is the perimeter of the triangle, in terms of b?
c. What is the equation?
d. How long is the base of the triangle?
e. How long is each of the equal sides of the triangle?

8. In a scalene triangle, the longest side measures 2 times that of the shortest side. The third side measures 5 inches more than the shortest side. The perimeter is 37 inches. How long is each side?

a. If the shortest side is x inches long, how long is the longest side, in terms of x?
b. How long is the third side, in terms of x?
c. What is the perimeter, in terms of x?
d. What is the equation? **e.** How long is each side?

9. John and Henry together have \$48. John has 3 times as much money as Henry. How much does each have?

a. If Henry has n dollars, how many dollars does John have, in terms of n?
b. How many dollars do they have together, in terms of n?
c. What is the equation?
d. How many dollars does Henry have?
e. How many dollars does John have?

10. Mary has saved $\frac{3}{5}$ of the money she needs to buy a coat. If she has saved \$12, how much will the coat cost?

a. If the coat will cost x dollars, how much has Mary saved, in terms of x?
b. What is the equation?
c. How much will the coat cost?

11. Martin and George purchased a boat for \$28. Martin paid \$4 more than George. How much did each of the boys pay?

a. If George paid n dollars, how much did Martin pay, in terms of n?
b. How much did both boys pay, in terms of n?
c. What is the equation?
d. How much did George pay?
e. How much did Martin pay?

Jane and Mary made sandwiches for the school picnic. Jane made 10 more sandwiches than Mary. Together they made 140 sandwiches. How many did each make?

EXAMPLE

Let x = the number of sandwiches Jane made (1)

$x - 10$ = the number of sandwiches Mary made (2)

$x + x - 10 = 140$ (Why?) (3)

$$\left.\begin{array}{rl} 2x - 10 = 140 & \text{or} \quad 2x + (^-10) = 140 \quad \text{Why?} \\ \text{Then } 2x + (^-10) + 10 = 140 + 10 & \\ 2x = 150 & \\ x = & ? \text{ the number of sandwiches Jane made} \\ x - 10 = & ? \text{ the number of sandwiches Mary made} \end{array}\right\} \quad (4)$$

How many did Jane make? How many did Mary make?

Check in the original word problem. (5)

Below are the Steps to follow in setting up an equation to solve a problem. In the above Example you will find in parentheses at the right of the statements numerals showing how each Step was used.

How to Use Equations to Solve Problems

Step 1. Express one of the numbers you need to find by a letter. Often we use x because it is easy to write and is not confused with any numeral. We call the letter we use a variable.

Step 2. Express the facts and relationships in the problem using the letter chosen.

Step 3. When you have two different expressions for the same quantity, set up an equation.

Step 4. Solve the equation.

Step 5. Check the answer by replacing the letters in the original problem by the answers you have found.

Let us see how these Steps are used in another Example.

EXAMPLE

Jim and Henry together had $5.30. Jim had 6¢ more than three times as much as Henry. How much money does each have?

Let us analyze this problem by using the five Steps above.

(1) We express the amount of money each has as a variable:
Let x = the number of cents Henry has.
(2) Then $3x + 6$ = the number of cents Jim has. Why?
(3) Then the number of cents both have is $4x + 6$. Explain.
Also, 530 = the number of cents both have.
Then $4x + 6 = 530$.

Complete Steps 4 and 5 by solving the equation and checking the answer. Practice each of these Steps in the problems that follow.

Follow the Steps carefully in solving the following problems.

1. The Andersons' house cost nine times as much to build as their garage. If both cost $20,000, how much did each cost?

2. Mr. Clark gave Ted and Sally their allowances. Sally received 3 times as much as Ted. Together their allowances total $3.60 a week. How much does each receive?

3. The perimeter of a rectangle is 64 feet. The length is 3 times the width. How long is each side? (Draw the figure and label the sides.)

4. The length of a rectangle is 4 inches longer than the width. If the perimeter is 28 inches, what are the dimensions? (Draw the figure.)

5. In a class of 33 pupils, there are 5 more girls than boys. How many of each are there?

6. Three times a number, increased by 16, is 64. Find the number.

7. Mary and Linda sold 15 boxes of Girl Scout cookies. Linda sold twice as many as Mary. How many did each sell?

8. Jim and his father had traveled 120 miles on a trip, each driving part of the way. Jim had driven 30 miles more than his father. How far did each drive?

9. The sum of three numbers is 120. The second number is 4 times the first, and the third is 7 times the first. Find the numbers.

10. Mabel and Helen have $40. Helen has $10 less than Mabel. How much has each?

11. A farmer owned 800 acres of land. He divided it among his three sons. The first received twice as many acres as the second, and the third 100 acres less than the other two together. How many acres did each receive?

12. The sum of three numbers is 600. The second number is twice the first and the third is equal to the first and second added together. What are the three numbers?

13. A number increased by twice itself gives 105. Find the number.

Some people like to think of an equation as a *balance*. Since both sides of a true equation name the same number, any operation may be performed on one side if it is also performed on the other side of the equation. This property can be illustrated by using a balance.

EXAMPLE

The length of a field is 47 rods. The length lacks 9 rods of being twice as long as it is wide. How wide is it?

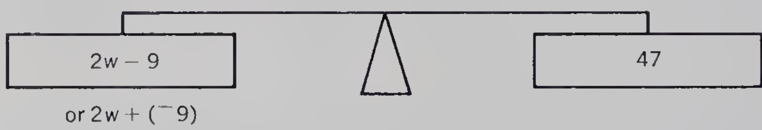
The relationship between the length, l, and width, w, can be expressed as:

$$2w - 9 = l$$

Since the length is 47 rods, the equation becomes:

$$2w - 9 = 47 \qquad \text{or} \qquad 2w + (^-9) = 47$$

Now think of the equation as being in balance.



The left side must always be balanced by the right side. Since we need to add 9 to the left side, we must also add the same number to the right side in order to keep the two sides in balance. This may be illustrated in this way:

Solve and check the following equations. Sketch a balance to illustrate your solution for each of the first four equations.

1. $x - 7 = 5$	**5.** $n - 5 = 4$	**9.** $n - 13 = 18$
2. $n - 9 = 16$	**6.** $x - 16 = 9$	**10.** $x - 5 = 16$
3. $x - 5 = 8$	**7.** $n - 5 = 12$	**11.** $x - 1.5 = 6.5$
4. $a - 8 = 60$	**8.** $x - 4 = 10$	**12.** $x - 4\frac{1}{3} = 5\frac{2}{3}$

Set up the equation and solve:

13. What number reduced by 19 is equal to 73?

14. If 27 less than a certain number is 56, what is the number?

15. If a certain number is decreased by 23, the result is 39. What is the number?

16. After Henry had spent \$15 for a pair of slacks and \$22 for a jacket, he still had \$37 left from his savings from last summer. How much did he save last summer?

The balance can also be used to illustrate the solution of an equation that involves subtraction. However, we have shown that adding the additive inverse of a number is the same as subtracting the number. For example:

$$n + 12 = 23$$

Then since $12 + (^-12)$ is the same as $12 - 12$,

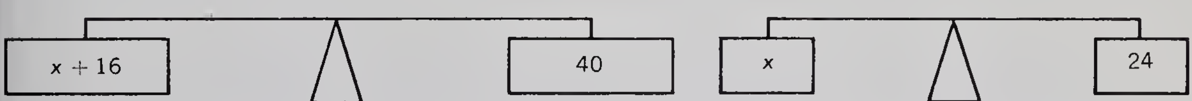
$$\begin{aligned} n + 12 + (^-12) &= 23 + (^-12) \\ n &= 11 \end{aligned}$$

EXAMPLES

1. A bag of sugar together with a 16-pound weight, balances a 40-pound weight on the other side. Find the weight of the sugar. Let x represent the weight of the sugar. Then the equation is:

$$x + 16 = 40$$

To solve for x, we can subtract 16 from both sides since $(16 - 16) = 0$, or we can add $(^-16)$ to both sides, since $16 + (^-16) = 0$. This is permissible, since the balance will be maintained.



2. Twice a certain number decreased by 5 is equal to 25 more than the number. What is the number?
Let n represent the number. Then the equation is:

$$\begin{array}{llrcl} & (1) & 2n - 5 &=& n + 25 \\ \text{or} & & 2n + (^-5) &=& n + 25 \\ \text{Then} & (2) & 2n + (^-5) + 5 &=& n + 25 + 5 \\ \text{or} & & 2n &=& n + 30 \\ \text{Then} & (3) & 2n + (^-n) &=& n + (^-n) + 30 \\ & & n &=& 30 \end{array}$$

Check: $(2 \times 30) - 5 = 30 + 25$
$60 - 5 = 55$

Illustrate the solution of each of the following:

17. $x + 24 = 29$
18. $n + 28 = 71$
19. $b + 8 = 10$
20. $x + 22 = 35$
21. $x + 14 = 20$
22. $n + 10 = 14$
23. $x + 9 = 25$
24. $n + 4 = 9$
25. $x + 6 = 15$
26. $n + 4 = 17$
27. $2n + 6 = 15$
28. $x + 3.4 = 9.2$

The balance can also be used to illustrate the use of the multiplicative inverse in solving equations.

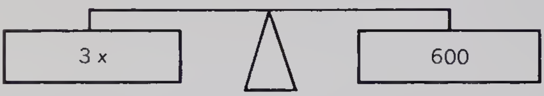
EXAMPLE

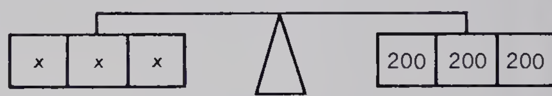
Henry helped his father pick strawberries last week. Together they picked 600 quarts. His father picked twice as many as Henry. How many quarts did Henry pick?

Let x = the number of quarts Henry picked.
Then $2x$ = the number of quarts his father picked.
And $x + 2x$ = the number of quarts both picked.
Since both picked 600 quarts, the equation is:

$$3x = 600$$

To obtain x alone on the left side, we must multiply both sides by the multiplicative inverse of the coefficient of x. This means that if the balance is to be maintained, both sides must be multiplied by $\frac{1}{3}$.





This gives us x on the left side, and 200 on the right side.

Sometimes the use of the multiplicative inverse is not so easy to illustrate. Suppose the equation is:

$$5.7x = 34.2$$

Then both sides are to be multiplied by the multiplicative inverse of 5.7, which is $\frac{1}{5.7}$.

$$\frac{1}{5.7} \times 5.7x = \frac{1}{5.7} \times 34.2$$

(Remember that $\frac{1}{5.7} \times 34.2$ is the same as $34.2 \div 5.7$.)

Then: $$x = 6$$

Solve each equation, stating the reason you are using at each step. Check each answer.

1. $9x = 117$
2. $18n = 162$
3. $16x = 256$
4. $25n = 225$
5. $13x = 273$
6. $24n = 360$
7. $17x = 85$
8. $14n = 126$
9. $16n = 400$

10. $6x = 714$

11. $15x = 2.25$

12. $25n = 22.5$

13. $15n = 120$

14. $3.2x = 1.92$

15. $1.8x = 24$

16. $5x - 15 = x + 5$

17. $4x + 2 = x + 23$

18. $6x - 4 = 4x + 2$

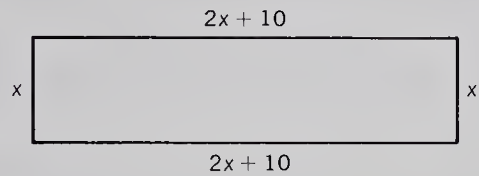
19. $5x + 25 = 2x + 31$

20. $15x - 2 = 10x + 18$

21. $24x - 36 = 12$

22. Five times a number decreased by 15 is equal to the number increased by 5. What is the number?

23. The length of a field is 10 rods more than twice its width. Its perimeter is 500 rods. What are its dimensions?

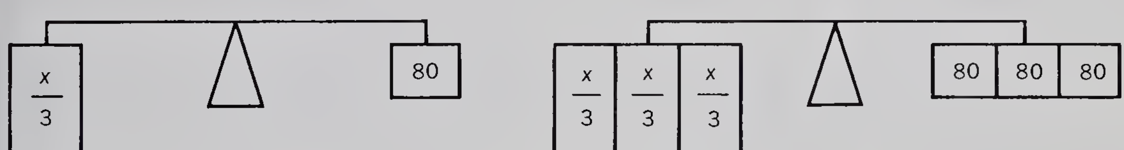


EXAMPLE

Last summer Jim was able to save $\frac{1}{3}$ of what he earned. His savings amounted to \$80. How much did he earn? Let x represent the amount Jim earned.

Here the equation is: $\frac{1}{3}x = 80$ or $\frac{x}{3} = 80$

To have the variable alone on the left side with a coefficient 1, we multiply by 3, the multiplicative inverse of $\frac{1}{3}$. To maintain balance, 80 must also be multiplied by 3.



Solve each equation, and check by substituting the value of the variable in the original equation.

24. $\frac{x}{2} = 6$

25. $\frac{5a}{7} = 15$

26. $\frac{a}{5} = 4$

27. $\frac{x}{7} = 11$

28. $\frac{a}{6} + 7 = 32$

29. $\frac{3x}{4} + 6 = 12$

30. $\frac{5a}{7} + 15 = 45 - a$

31. $\frac{2x}{3} + 12 = 42 - x$

32. $\frac{7x}{2} + 14 = 2x + 20$

33. $\frac{5x}{16} = 42 - x$

34. $8x - 6 = 34$

35. $4x = x + 18$

36. $2x - 7 = x$

37. $5x - 6 = x + 10$

38. $2x + 16 = 30$

39. $2.5x + 13 = 25$

40. $\frac{3x}{4} + 9 = \frac{x}{4} + 13$

41. $3.5x + 6 = x + 11$

42. $2\frac{1}{2}r = 3 - r$

43. $2x - 9 = 11 - 2x$

Set up and solve the equation for each of the following:

44. If 7 is added to three times a number, the result is 25. What is the number?

45. If $\frac{2}{3}$ of a number is increased by 4, the sum is 26. What is the number?

46. If 4 times a number is decreased by 5, the result is the same as if twice the number were increased by 11. What is the number?

47. If twice a certain number is increased by 7, the result is the same as if 3 times the number were subtracted from 27. What is the number?

48. The length of a room is 4 feet more than twice the width. Its perimeter is 68 feet. What are the dimensions of the room?

49. The perimeter of a field is 312 rods. The width is 6 feet more than $\frac{2}{3}$ of the length. What are the dimensions of the field?

Oral Exercises

If x represents any positive or negative number or zero, express:

1. 7 more than the number
2. 5 times the number
3. The number decreased by 9
4. 4 more than twice the number
5. 8 less than twice the number
6. $\frac{3}{4}$ of the number
7. $\frac{1}{2}$ of the number decreased by 6
8. 10 more than $\frac{1}{4}$ of the number
9. 6 less than 15 times the number
10. 25 more than 12 times the number
11. A field is x rods wide, and the length is 10 rods more than the width. Express its length in terms of x.

12. A triangle has sides of x, $x + 2$, and $x + 5$. Express its perimeter in terms of x.

13. Mary is x years old, and Helen is 7 years older. Express Helen's age in terms of x.

If x represents any positive or negative number or zero, tell in words what each of the following expressions means.

14. $x + 15$
15. $x - 7$
16. $2x$
17. $3x + 2$
18. $4x - 9$
19. $\frac{x}{2}$
20. $\frac{3x}{4}$
21. $5x^2 + 1$
22. $\frac{x}{3} + 7$
23. $\frac{2x}{5} - 12$
24. $\frac{3x}{7} - \frac{5}{2}$
25. $x + \frac{x}{3}$
26. $2x - \frac{x}{2}$
27. $\frac{x - 3}{2}$
28. $\frac{2x + 5}{3}$
29. $x^2 + x$
30. $x^2 + 4$
31. $\frac{x^2}{4}$
32. $\frac{x^2}{x}$

The following are problems you should be able to solve without the use of a pencil. See how well you do, writing only the answer.

33. Four more than a number is 7. Find the number.
34. A number decreased by 3 gives 10. Find the number.
35. Twice a number is 24. What is the number?
36. If one of two numbers is 7 and their sum is 12, find the other.
37. If a number is doubled it becomes 48. What is the number?
38. Half of a number is 14. Find the number.
39. A number increased by 6 gives 28. Find the number.
40. If 66 is three times a number, what is the number?
41. Four times a number is 80. Find the number.
42. Two numbers total 56. One is 20. What is the other?
43. If three numbers total 15 and two of them are 4 and 5, what is the third number?
44. One-third of a number is 6. What is the number?
45. A number decreased by 7 gives a result of 43. What is the number?
46. Fives times a certain number gives a result of 60. What is the number?
47. One of two numbers is 11. Their sum is 34. What is the other number?
48. The distance around a square is 44 inches. What is the length of a side of the square?

Steps in Using Equations to Solve Problems

The following Steps are intended as an aid in solving problems that involve equations.

Step 1. Use a letter to stand for the smaller of the numbers to be found.
Step 2. Express the other numbers in the problem by using the same letter.
Step 3. Find a way of expressing some quantity both as a letter and as a numeral, and set up a true equation using these expressions.
Step 4. Solve the equation.
Step 5. Check the answer.

Use these Steps in solving these problems.

1. The sum of two numbers is 188. One number is three times as great as the other. What are the numbers?

2. There are 44 more girls than boys in the Wilson High School. The total enrollment is 376. How many girls and how many boys are there?

3. Henry has a 12-ft. board which he wishes to saw so that one piece will be 2 ft. longer than the other. How long should each piece be?

4. If a certain number is multiplied by 3, the result is the same as if the number were increased by 12. What is the number?

5. The perimeter of a rectangular field is 480 rods. The field is twice as long as it is wide. What are its dimensions?

6. Five times a certain number decreased by 35 is equal to twice the number increased by 10. What is the number?

7. If a certain number is multiplied by 3, the result is the same as if 25 were added to one-half the number. What is the number?

8. In a certain factory a machinist's weekly earnings are $5 less than twice as much as those of his helper. Together they earn $265. How much does each earn?

9. If one-half of a number is increased by 20, the result is the same as if twice the number were increased by 2. What is the number?

10. If one-half of a number is added to three times the number, the result is the same as if 18 were subtracted from five times the number. What is the number?

11. John's father is 27 years older than John. If he is just four times as old as John, how old is each of them?

The use of algebraic language makes it possible to examine some important operations in arithmetic and to identify some properties of these operations that you have been using and that were discussed in Chapter 3 with regard to the set of whole numbers. We are concerned here with the set of positive and negative numbers and zero.

Suppose you are asked to solve:

$$24 + 13 + 37 = n$$

You cannot solve it in one step since you can add only two numbers in any one step. Which pair of addends should you combine first?

You can proceed in either of two ways as shown by the parentheses, which indicate operations to be performed first:

A	B
$(25 + 13) + 37 = n$	$25 + (13 + 37) = n$
$38 \quad + 37 = n$	$25 + \quad 50 \quad = n$
$75 = n$	$75 = n$

Remember! The fact that you get the same answer by grouping addends in either of the ways illustrated above illustrates the *associative property* of addition. Stated in general terms:

If a, b, and c represent any positive or negative numbers or zero, then

$$(a + b) + c = a + (b + c)$$

Does the associative property hold under subtraction? That is, is this a true statement?

$$(16 - 8) - 4 = 16 - (8 - 4)$$

The operations within the parentheses are to be performed *first*. Try several other examples to see if the property holds for any of them.

Another significant property of addition that we tend to take for granted is the *commutative property*. You know that if you have \$5 and earn \$4 you will then have \$9. It is also true that if you start with \$4 and earn \$5 you will have \$9. That is, $4 + 5 = 5 + 4$. Stated in general terms:

If a and b represent any positive or negative numbers or zero, then

$$a + b = b + a$$

Does the commutative property hold under subtraction? That is, is this statement true?

$$a - b = b - a$$

Try several other Examples.

Frequently one or both of these properties may be used to simplify an addition operation.

EXAMPLE

Solve: $22 + 39 + 28 = n$	
$22 + 28 + 39 = n$	Commutative property
$(22 + 28) + 39 = n$	Associative property
$50 + 39 = n$	
$89 = n$	

In the following problems, perform the operations as indicated, first performing the operation within the parentheses. Then use the properties of addition to simplify the operation.

1. $26 + (24 + 18)$

2. $52 + (25 + 18)$

3. $(18 + 48) + 12$

4. $(19 + 59) + 21$

5. $(13 + 48) + 7$

6. $29 + (56 + 11)$

7. $\frac{3}{4} + (\frac{5}{8} + \frac{1}{4})$

8. $(\frac{3}{16} + \frac{3}{8}) + \frac{13}{16}$

9. $1\frac{1}{5} + (\frac{9}{10} + \frac{4}{5})$

10. $2\frac{4}{7} + (\frac{1}{2} + \frac{3}{7})$

11. $(6\frac{7}{8} + \frac{5}{6}) + \frac{1}{6}$

12. $5\frac{2}{3} + (\frac{3}{4} + 1\frac{1}{3})$

Following are problems of addition involving several addends. Get the sums by using the properties of addition.

13. $51 + 38 + 49 + 62 + 100$

14. $25 + 30 + 25 + 60 + 10$

15. $101 + 28 + 99 + 50 + 22$

16. $150 + 50 + 63 + 7 + 30$

17. $88 + 75 + 25 + 12 + 200$

18. $73 + 40 + 7 + 60 + 20$

19. $58 + 40 + 2 + 31 + 19$

20. $22 + 14 + 36 + 28 + 50$

21. $17 + 60 + 23 + 100 + 10$

22. $15 + 70 + 15 + 200 + 300$

23. $85 + 60 + 15 + 30 + 10$

24. $110 + 28 + 90 + 22 + 50$

25. $86 + 22 + 14 + 50 + 28$

26. $17 + 13 + 70 + 25 + 75$

27. $64 + 80 + 16 + 35 + 45$

28. $100 + 87 + 50 + 13 + 50$

29. $91 + 91 + 9 + 9 + 500$

30. $730 + 250 + 70 + 250$

31. $86 + 55 + 14 + 50 + 45$

32. $115 + 75 + 85 + 25 + 100$

33. $19 + 40 + 81 + 60 + 300$

34. $44 + 20 + 16 + 80 + 15$

35. $77 + 7 + 23 + 93 + 75$

36. $400 + 200 + 600 + 800$

Do you recall if the associative and commutative properties hold under multiplication? If the associative property holds, then, for example:

$$(14 \times 16) \times 9 = 14 \times (16 \times 9)$$

Carry out the multiplication on both sides of the equals sign, and see if this is the case. In general:

If a, b, and c are any positive or negative numbers or zero,

$$a \times (b \times c) = (a \times b) \times c$$

You have made use of the commutative property in multiplication computations, particularly in Chapter 3. If you are to find the product of 7 and 525, you will use the form

$$\begin{array}{r} 525 \\ \underline{7} \end{array} \quad \text{rather than} \quad \begin{array}{r} 7 \\ \underline{525} \end{array}$$

knowing that both have the same product. In general:

$$a \times b = b \times a$$

1. Does the associative property hold under division? For example, is this statement true?

$$(12 \div 4) \div 3 = 12 \div (4 \div 3)$$

Try several similar examples.

2. Does the commutative property hold under division? Try several examples before you answer.

In some cases where you are to find the product of several factors it may make a difference in difficulty which multiplication you perform first. For example:

$$4 \times 14 \times 25 = n$$

A	B
$(4 \times 14) \times 25 = n$	$14 \times (4 \times 25) = n$
$56 \quad \times 25 = n$	$14 \times \quad 100 \quad = n$
$1400 = n$	$1400 = n$

Note that in B we used both the commutative and associative properties to simplify the operation.

The following operations are to be performed as given, with factors inside the parentheses multiplied first. Then use the above properties to simplify the computation.

3. $25 \times (4 \times 9)$
4. $(8 \times 7) \times 125$
5. $20 \times (19 \times 5)$
6. $(4 \times 13) \times 250$
7. $(15 \times 2) \times 50$
8. $(5 \times 13) \times 20$
9. $250 \times (17 \times 4)$
10. $(8 \times 21) \times 125$
11. $(5 \times 17) \times 20$
12. $(4 \times 9) \times 25$
13. $2 \times (21 \times 5)$
14. $(9 \times 5) \times 20$

Sometimes one or both of the factors can be factored and the properties of multiplication used to simplify the operation.

EXAMPLE

Solve:

	$36 \times 75 = n$	
	$9 \times 4 \times 3 \times 25 = n$	Factoring
	$9 \times 3 \times 4 \times 25 = n$	Commutative property
	$(9 \times 3) \times (4 \times 25) = n$	Associative property
	$27 \times 100 = n$	
	$2700 = n$	

Factor and find the products.

15. 16×25
16. 36×50
17. 25×32
18. 24×125
19. 12.5×32
20. 24×75
21. 16×375
22. 2.5×36
23. 37.5×24
24. 75×12
25. 625×4
26. 1.25×56
27. 18×50
28. 44×2.5
29. 48×75
30. 125×8
31. 56×25
32. 44×75
33. 48×75
34. 2.5×24
35. 75×32
36. 16×125
37. 88×25
38. 55×20
39. 75×28
40. 150×12
41. 7.5×44
42. 88×75
43. $44 \times 50 \times 3$
44. $16 \times 25 \times 7$
45. $80 \times 25 \times 6$
46. $25 \times 16 \times 12$
47. $28 \times 50 \times 2$
48. $17 \times 50 \times 4$
49. $32 \times 12.5 \times 8$
50. $75 \times 40 \times 12$
51. $16 \times 75 \times 12$
52. $56 \times 25 \times 8$
53. $125 \times 4 \times 16$
54. $75 \times 16 \times 50$

The distributive property of multiplication and division is another property that you have used frequently. (See Chapter 3.) Examine the following:

$$7 \times 253$$

This may also be expressed as:

$$7 \times (200 + 50 + 3)$$

The distributive property with respect to addition may be described in general terms as follows:

If a, b, c, and d represent any positive or negative numbers or zero, then

$$a \times (b + c + d) = ab + ac + ad$$

Let us see how this works using numbers.

$$\begin{array}{rl} 253 & \\ \underline{7} & \\ 21 & = 7 \times 3 \\ 350 & = 7 \times 50 \\ \underline{1400} & = 7 \times 200 \\ 1771 & \end{array}$$

In other words, while you do not actually set it all down, when you multiply 7×253 you use the distributive property to find $(7 \times 3) + (7 \times 50) + (7 \times 200)$.

This property is frequently useful in performing multiplications without the use of pencil and paper.

EXAMPLES

1. Find: $8 \times 83 = ?$

$$\begin{aligned} 8 \times (80 + 3) &= (8 \times 80) + (8 \times 3) \\ &= \quad 640 \quad + \quad 24 \\ &= 664 \end{aligned}$$

2. Find: $9 \times 5\frac{1}{2} = ?$

$$\begin{aligned} 9 \times 5\tfrac{1}{2} &= (9 \times 5) + (9 \times \tfrac{1}{2}) \\ &= \quad 45 \quad + \quad 4\tfrac{1}{2} \\ &= 49\tfrac{1}{2} \end{aligned}$$

Use the distributive property to find the products on page 278.

1. 9×35	**4.** 7×5.3	**7.** $\frac{1}{2} \times 4\frac{1}{2}$
2. 8×16	**5.** 8×47	**8.** 5×6.7
3. 6×5.4	**6.** $6 \times 8\frac{1}{2}$	**9.** $6 \times 3\frac{2}{3}$

Multiplication is also distributive with respect to subtraction. That is:

If a, b, and c represent any positive or negative numbers or zero, then

$$a \times (b - c) = ab - ac$$

EXAMPLE

Find: $5 \times 95 = ?$

$$\begin{aligned} 5 \times 95 &= 5 \times (100 - 5) \\ &= (5 \times 100) - (5 \times 5) \\ &= \quad 500 \quad - \quad 25 \\ &= 475 \end{aligned}$$

Use the distributive property with respect to subtraction to find these products.

10. 3×47	**13.** 4×49	**16.** 8×45
11. 5×39	**14.** 6×99	**17.** 9×98
12. 8×98	**15.** 7×29	**18.** 7×95

Division is also distributive with respect to addition. That is:

If a and b represent any positive or negative numbers or zero, and c represents any positive or negative number, then

$$(a + b) \div c = (a \div c) + (b \div c)$$

Note that c cannot be equal to zero since division by zero is meaningless.

EXAMPLE

Divide: $245 \div 7 = ?$

245 is renamed as $(210 + 35)$.

Then:

$$\begin{aligned} (210 + 35) \div 7 &= (210 \div 7) + (35 \div 7) \\ &= \quad 30 \quad + \quad 5 \\ &= 35 \end{aligned}$$

We used 210 as the first of the two numbers into which 245 was renamed because it is the largest multiple of (7×10) that can be taken from 245. Divide 245 by 7, and you will see why 245 was renamed as it was.

With a larger dividend you can work with a multiple of 100.

EXAMPLE

Divide: $1896 \div 8 = ?$

1896 is renamed as $(1600 + 240 + 56)$.

Explain why 1600 and 240 are selected as two of the numbers into which 1896 is renamed. Complete the exercise.

Use the distributive property to perform these divisions. Do as little written work as possible.

19. $384 \div 6$
20. $234 \div 9$
21. $532 \div 7$
22. $3648 \div 8$
23. $2604 \div 7$
24. $1386 \div 6$
25. $8552 \div 8$
26. $1584 \div 11$
27. $1808 \div 8$
28. $1524 \div 6$
29. $1585 \div 5$
30. $2478 \div 7$

Both under multiplication and division, the distributive property is very useful in working with variables as well as with constants.

EXAMPLES

1. Find: $6(2a + 7)$

$$\begin{aligned} 6(2a + 7) &= (6 \times 2a) + (6 \times 7) \\ &= 12a + 42 \end{aligned}$$

2. Find: $(8n + 32) \div 4$

$$\begin{aligned} (8n + 32) \div 4 &= (8n \div 4) + (32 \div 4) \\ &= 2n + 8 \end{aligned}$$

Use the distributive property to express each of the following:

31. $7(a + 8)$
32. $5(2n - 3)$
33. $8(3y - 2)$
34. $(12a + 6) \div 6$
35. $9(2a + 6)$
36. $(9x + 18) \div 3$
37. $(6x - 8) \div 2$
38. $9(3y + 5)$
39. $(16a + 12) \div 4$
40. $(15y + 20) \div 5$
41. $(36n + 18) \div 9$
42. $8(7x - 2)$
43. $(5x - 12)6$
44. $\frac{1}{2}(6x - 14)$

The following Example illustrates the use of the distributive property in solving equations.

EXAMPLE

The sum of three numbers is 26. The second number is two more than the first, and the third is twice the second. What are the numbers?

Express the relationships among the numbers in algebraic language:

Let x = the first number,
and $x + 2$ = the second number,
then $2(x + 2)$ = the third number.

Then the equation is:

	$x + (x + 2) + 2(x + 2) = 26$	The coefficient of $(x + 2)$ is 1.
or	$x + x + 2 + 2x + 4 = 26$	Distributive property
Then	$4x + 6 = 26$	Combining variables and combining constants.

Complete the solution.

In the setting up of the equation and the solution you can see how parentheses may be convenient in expressing relationships, and how the distributive property of multiplication is used in removing the parentheses before you solve the equation.

Notice also how it was necessary to combine variables and to combine constants before undertaking the solution of the equation. When they have been combined there should not be more than two terms on either side of the equation.

Follow the procedure illustrated above to solve these equations.

1. $2(x + 5) = 14$

2. $3x = 2(x + 2)$

3. $7x = 3(x - 5) + 22$

4. $5(x + 3) = 25$

5. $5(x + 3) + 17 = 42$

6. $27 = 3(x + 3)$

7. $9(x + 1) = 12$

8. $6(x - 7) = 6$

9. $5x = 2(x + 4)$

10. $6x - 9 = 3(x + 4)$

11. $3(x + 7) = 228$

12. $2x + 3(x + 5) = 4(x + 8)$

13. $7x + 2(x + 5) = 3(2x + 7)$

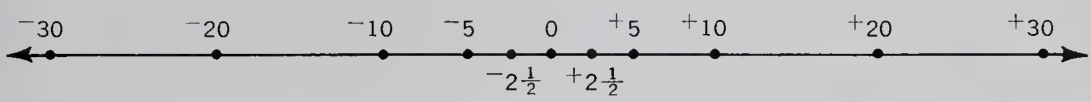
14. $4(x - 7) = 3(x - 2)$

15. $3(x - 2) + 7 = 2(x + 2)$

16. $7(2x + 1) - 7 = 28$

THE SET OF RATIONAL NUMBERS

Among the most important purposes for studying mathematics is the gaining of an understanding of the number system and learning about the uses of new numbers. In this chapter we have learned about signed numbers, and that each positive number has a "mate" that is equal to it in absolute value, but is negative in sign, and hence is associated with a point on the number line that is to the left of zero. The set of numbers that we can now represent on the number line includes several subsets, some of which we have studied, and others that we have used without learning their names.



1. On which side of zero do points associated with the set of natural numbers lie?
2. Does the set of whole numbers include zero?
3. The set of natural numbers is a subset of the set of whole numbers. What number is a member of the set of whole numbers that is not a natural number?
4. The set of *integers* includes the whole numbers and their negative "mates." The set of *positive integers* is identical with the set of natural numbers. The *negative integers* are the negative "mates" to the positive integers. What whole number is not included with the positive or negative integers?
5. Since zero is also an integer, what three subsets make up the set of integers?
6. The set of fractional numbers of arithmetic have their "mates" (the set of negative fractional numbers) associated with points to the left of zero on the number line. These together with zero make up the set of *rational numbers.* They are defined as the set of numbers that can be expressed in the form $\frac{a}{b}$, $b \neq 0$, where a and b are integers. What do you call the subset of the set of rational numbers when $b = 1$? Why is $b \neq 0$?
7. What is another name for the set of natural numbers?
8. What is another name for fractional numbers?
9. Are integers also rational numbers?
10. Is zero a member of the set of rational numbers?
11. Is this statement true? The set of rational numbers is a subset of the set of integers.

USING FORMULAS TO SOLVE PROBLEMS

When you use a formula to solve a problem you substitute numerals for letters. The letters might stand for any number. For example, in the formula $A = lw$, the length of the rectangle might be 1 inch, 2 feet, $3\frac{1}{2}$ yards, or 4.7 miles, or any other measurement. As we have already stated, the letters in a formula are called *variables*, or *pronumerals*. Do you see a similarity between pronumerals and pronouns? A pronumeral, or variable, is a symbol that represents a number.

People who work with formulas have found that they can work more efficiently and accurately by following these Steps:

Step 1. Choose the formula you intend to use and write it down.
Step 2. Beneath the formula write what you know about each letter in the formula.
Step 3. Rewrite the formula, substitute the numerals for the letters.
Step 4. Perform the indicated computations.

EXAMPLE

Find the area of a triangle whose base measures 6 in. and whose height is 5 in.

Step 1. $A = \frac{1}{2}bh$

Step 2. $\begin{cases} A = ? \\ b = 6 \text{ in.} \\ h = 5 \text{ in.} \end{cases}$

Step 3. $A = \frac{1}{2} \times 6 \times 5$

Step 4. $A = 15$ sq. in.

In each of the following exercises, first state the rule expressed by the formula. Then follow the four Steps to complete the computation.

1. $A = \frac{1}{2}bh$; $b = 18$ inches, $h = 10$ inches
2. $A = \pi r^2$; $\pi = 3.14$, $r = 3$ feet
3. $C = \pi d$; $\pi = 3.14$, $d = 6$ inches
4. $p = 2l + 2w$; $l = 12$ yards, $w = 7$ yards
5. $d = 2r$; $r = 5\frac{3}{8}$ inches
6. $p = a + b + c$; $a = 6$ inches, $b = 7$ inches, $c = 8$ inches
7. $d = rt$; $r = 40$ miles per hour, $t = 4\frac{1}{2}$ hours
8. The perimeter of a rectangle (P) equals twice the length (l) plus twice the width (w). What is the perimeter of a rectangle whose length is 25 inches and whose width is 14 inches? If the area (A) of a rectangle is found by the formula $A = l \times w$, find the area.

Set up a formula to express each of the following rules. (The letters in parentheses are suggested for your use in the formula.) Then use the formula to solve the problem that follows it.

9. The average rate of speed (r) while making a trip equals the distance traveled (d) divided by the number of hours (t).
A salesman drove 4 hours and traveled 168 miles. What was his average hourly speed?

10. The perimeter (p) of a triangle equals the sum of the measures of the three sides (a, b, and c).
Find the perimeter of a lot shaped like a triangle with sides measuring 75 feet, 166 feet, and 152 feet.

11. The circumference (C) of a circle equals pi (π) times the measure of a diameter (d). Find the circumference of the circle with a radius of 16 inches, if $\pi = 3.14$.

12. The area (A) of a triangle equals $\frac{1}{2}$ the measure of the base (b) times the measure of the altitude (h). The altitude of a triangle measures 6 inches and its base measures 4 inches. What is its area?

13. The price (p) of a certain number of items (n) priced at (d) dollars each equals the cost of each item multiplied by the number of items. Find the cost of 6 shirts which sell for $2.75 each.

14. The volume (V) of a rectangular prism equals the length of the base (l) times the width of the base (w) times the height (h).
Find the volume of a box that is 18 inches long, 12 inches wide, and 10 inches high.

15. The number of fruit trees (t) in an orchard equals the number of trees (n) in one row multiplied by the number of rows (r).
How many fruit trees are there in an orchard that has 24 trees in a row and has 16 rows?

16. The profit (p) made by a merchant equals his total sales (s) minus the sum of his costs (c) and expenses (e).
The total sales of a dress shop in November were $8675. The cost of the goods sold was $5205.15. The expenses of running the shop for the month totaled $2165.40. What was the profit for November?

17. The area of a trapezoid is found by multiplying the sum of the measures of the bases (a and b) by one-half of the height (h). The bases of a trapezoid measure 18″ and 24″ and the height is 16″. What is the area?

Formulas are very helpful in aiding our ability to solve problems easily and quickly as you have already seen in Chapter 5. However, sometimes the formula may not be written in the form we would find

most useful. For example, the formula for the perimeter of a rectangle is $p = 2l + 2w$ where p is the perimeter, l is the length, and w is the width. Now suppose we know the perimeter and length of the rectangle and want to find the width. The formula can be rewritten to solve for w by applying the methods you used to solve equations.

EXAMPLES

1. $p = 2l + 2w$ Subtract $2l$ from both sides, or add ^-2l.
   $p - 2l = 2w$ Divide both sides by 2, or multiply both sides by $\frac{1}{2}$.
   Then,
   $$\frac{p-2l}{2} = w$$
   or,
   $$w = \frac{p-2l}{2}$$

2. What is the width of a rectangle if the perimeter is 40 inches and the length is 15 inches?
   $$w = \frac{p-2l}{2} \qquad p = 40;\ l = 15$$
   $$w = \frac{40 - (2 \times 15)}{2}$$
   $$w = \frac{40-30}{2}$$
   $$= \frac{10}{2} = 5$$
   Therefore the width = 5 inches.

Rearrange the following formulas by solving for the specified variable.

18. The formula for the circumference of a circle is $C = 2\pi r$ ($\pi = 3.14$). Solve the formula for r. What is the radius of a circle if its circumference is 68 inches? (Round your answer to the nearest hundredth.)

19. $A = \frac{1}{2}bh$ is the formula for the area of a triangle. Solve the formula for h (height) and find the height of a triangle if its area is 68 square inches and its base measures 20 inches.

20. $F = \frac{9}{5}C + 32$ is the formula for changing temperature from Celsius degrees to Fahrenheit degrees. Solve for C.

21. The boiling point of water on the Fahrenheit scale at sea level is 212°. Using the new formula obtained from Problem 3, what is the boiling point of water on the Celsius scale?

1. You will find it convenient, in working problems on transportation and travel, to use the formula $d = rt$, in which
   d is the distance (miles, feet, etc.)
   r is rate (miles per hour, feet per second, etc.)
   t is the time
   State $d = rt$ as a rule.
2. At a speed of 37 miles per hour, how far will a car travel in 2 hours?
3. Sound travels at about 1100 ft. per second. How far will sound travel in 5 seconds?
4. A jet plane has a speed of 550 miles per hour. How far can it travel in 4 hours?
5. A car is traveling at 50 miles per hour. How long will it take to travel 175 miles?

   HINT: You will find it convenient to use $t = \frac{d}{r}$.

   How is $t = \frac{d}{r}$ derived from $d = rt$?
6. A plane has a speed of 350 miles per hour. How long will it take to travel 2275 miles?
7. A truck makes a trip of 117 miles in $3\frac{1}{4}$ hours. What is the average hourly rate?
8. The Boy Scout Troop made a hike of 14 miles in 4 hours. What was its average rate?
9. A train travels from New York to Chicago in 15 hours. The distance is 960 miles. What is the average speed of the train?
10. A truck driver starts out at 10:30 A.M. to deliver a load to a city 240 miles distant. He is expected to arrive there by 4:30 P.M. What speed must he average in order to be on time?
11. A jet plane travels from Los Angeles to New York in $4\frac{1}{2}$ hours. The distance is 2400 miles. What is the average speed of the plane?
12. A plane travels from San Francisco to Portland in $1\frac{1}{2}$ hours. The distance is 720 miles. What is the average speed of the plane?
13. Mr. Johnson traveled a distance of 140 miles by automobile in $3\frac{1}{2}$ hours. What was his average speed?
14. How long will it take to travel $13\frac{1}{2}$ miles at an average speed of $3\frac{1}{2}$ miles an hour?
15. How far will an airplane travel in $6\frac{1}{2}$ hours at an average speed of 670 miles per hour?
16. How many feet per second is 60 miles per hour?

The speed or velocity of falling objects is described by the formula: Velocity = acceleration of gravity × time, or $V = gt$. The acceleration of gravity (g) is approximately 32 feet per second each second, and t stands for the time in seconds that an object has fallen.

EXAMPLE

A ball falling off the roof of an apartment house struck the ground in two seconds after it began to fall. What was the velocity of the ball as it struck the earth?

Using the formula $V = gt$

We know $g = 32$ and $t = 2$ seconds

Thus $V = 32 \times 2 = 64$ feet per second (approximately 44 miles per hour)

17. The formula $V = gt$ does not take into consideration the effect of the resistance of the air on a falling object. How does air affect the speed of a falling object? Does it affect a lightweight object more than a heavy object? In the following problems, disregard resistance of air.

18. Jack and Tom used a stop watch to find how long it took a rock falling off a cliff to strike the earth. The time was $3\frac{1}{2}$ seconds. What was the speed of the rock as it hit the earth?

19. What would be the speed of an object dropped from a plane if it hit the earth after falling 10 seconds?

20. In the formula $V = gt$, the velocity (V) is given in feet per second. Change your answer to Exercise 19 to miles per hour.

21. Scientists have also developed a formula to determine the distance a body will fall in a given time. This formula is Distance (or space) $= \frac{1}{2}$ acceleration of gravity × seconds × seconds, or $S = \frac{1}{2}gt^2$. Use this formula to find how far an object will fall in 8 seconds.

22. Using the formula in Problem 21, how far would an object fall in the time of one minute?

23. We know of course that some objects will fall faster than others. In our atmosphere, which will fall faster, a marble or a feather?

24. If it takes an object, which is falling from a private plane 8.7 seconds to hit the ground, what is the altitude of the plane? Round your answer to the nearest hundred feet.

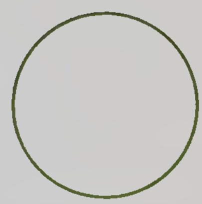
25. The object in Problem 24 is traveling at a speed of approximately 200 miles per hour when it hits the ground. Use the formula in Problem 20 to verify the speed of the object.

Your ability to solve mathematical problems will be improved by special practice in giving attention to all the available facts, and the relationships among them. These exercises are designed to help you to see how this is done and how important it is in problem solving to give careful attention to the data provided with the problem.

STEPS FOR SOLVING MATHEMATICAL PROBLEMS

1. Understand the problem.
2. Analyze the data.
3. Discover new facts.
4. Follow up and verify promising leads.
5. Review your solution.

Frequently you will find that you are prevented from finding the solution by "imaginary" restrictions that do not actually exist, which you yourself impose from force of habit.



For example, cut the pie (Figure at the right) into 8 pieces by making only 3 cuts. You are not allowed to arrange the pieces in layers between cuts.

Experiment, and see if you can find the answer, before reading any further.

Now examine your work. Did you assume that the pieces must be equal in size? No such restriction was made.

Did you improperly assume that all 3 cuts must be straight lines?

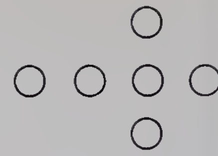
Try this: Cut the pie into four pieces by cutting along two diameters that are perpendicular to each other. You have used two cuts.

Now for the third cut, use a radius that measures less than a radius of the pie (say about two-thirds of the measure of a radius of the pie), and construct a circle whose center is the center of the pie. The circle will cross the cuts made by the perpendicular diameters. Thus, you will have cut the pie into 8 pieces.

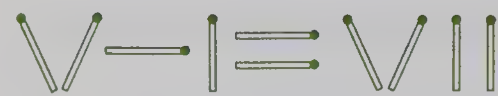
The problems that follow will give you a chance to exercise your creative ability. Do not give up too quickly on any problem. Be careful each time to see that you are not imposing unnecessary restrictions on the conditions of the problem.

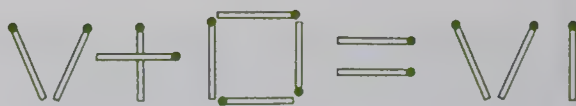
1. Six coins are arranged as at the right on the next page. You are

to move just one coin to a position such that each row, horizontal and vertical, contains four coins.



2. In each of the Illustrations below, the false equation represented by the matchsticks in each Figure can be made true by moving just one match. How?





3. Draw a circle as in Figure 1 below. Divide this circle into as many parts as possible by drawing four straight lines.

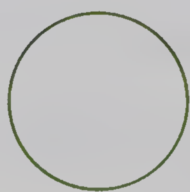


Figure 1

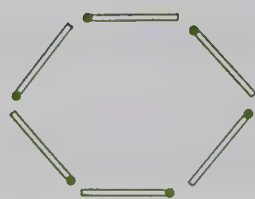
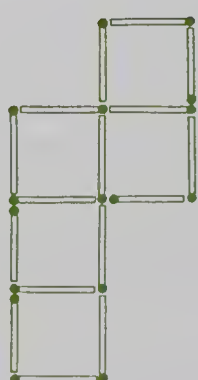


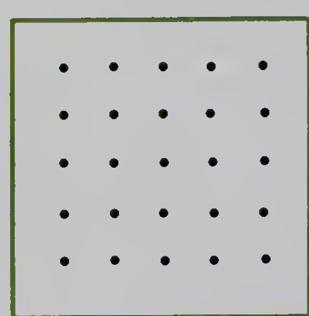
Figure 2

4. By moving two of the matches that form the shape of a hexagon (Figure 2 above), and adding one more, you can form two diamonds. Show how it is done.

5. By moving just two of the matches in the Figure below on the left, you can form four squares, instead of five, each touching one or two others. Show how it is done.

6. Copy the Figure below on the right. Connect 12 of the 25 dots in such a way as to form a cross which has 5 dots enclosed in it, and 8 dots outside it.





7. Suppose you had a drawer of socks which were identical except that some were white and the rest black. How many socks would you have to remove from the drawer before you were sure of a matching pair?

1. In the expression $\frac{3.5x}{4} - 9$:

a. What is the coefficient of the variable?
b. What is the multiplicative inverse of the coefficient of the variable?
c. What is the constant?
d. What is the additive inverse of the constant?
e. What is the absolute value of the constant?
f. If $x = 16$, what is the value of the expression?

2. If a, b, and c represent any rational numbers, which of the properties of the operations does each of the following equations illustrate?

a. $a \times (b \times c) = (a \times b) \times c$
b. $a \times (b + c) = ab + ac$
c. $a + b = b + a$
d. $(a + b) \div c = (a \div c) + (b \div c)$
e. $a \times (b - c) = ab - ac$
f. $(a + b) + c = a + (b + c)$

3. Some of the following statements are true and some are false. On your paper write T after the letter indicating those that are true. Write a corrected statement for each statement that is false.

a. The sum of a rational number and its additive inverse is zero.
b. On the number line, the distance from zero to a point associated with a number is the same as the distance to its additive inverse.
c. The point associated with any non-zero rational number is on the opposite side of zero from that of its additive inverse.
d. The absolute value of a rational number is equal to the absolute value of its additive inverse.
e. For subtraction of signed numbers, you may substitute the equivalent operation: Add the additive inverse of the subtrahend to the minuend.
f. For division with rational numbers you may substitute the equivalent operation: Multiply the dividend by the multiplicative inverse of the divisor.
g. The absolute value of $^-8$ is greater than the absolute value of $^+4$.

If a represents any rational number:

h. $+(^-a) = {}^-a$
i. $-(^-a) = {}^-a$
j. $-(^+a) = {}^-a$
k. $+(^+a) = {}^-a$

4. Copy each of the following pairs, and insert < or > to make a true statement.

a. $^-12$, $^+11$
b. $^+\frac{1}{8}$, $^-\frac{1}{2}$
c. $^-20$, 0
d. $^-18$, $^-30$
e. $^+\frac{12}{32}$, $^+\frac{11}{16}$
f. $^+8$, $^-2$

Part One

A. What property does each of the following statements illustrate?

1. $376 \times 892 = 892 \times 376$
2. $15 \times (8 \times 9) = (15 \times 8) \times 9$
3. $(60 + 25) \div 5 = (60 \div 5) + (25 \div 5)$
4. $12 \times (9 \times 15) = (12 \times 9) \times 15$
5. $63 \times (11 + 6) = (63 \times 11) + (63 \times 6)$
6. $8 \times (16 - 5) = (8 \times 16) - (8 \times 5)$

B. What must be the value for n to make each of these statements true?

1. If $12 \times n = 12 \times 7$, $n = ?$
2. If $12 \times (15 + n) = (12 \times 15) + (12 \times 9)$, then $n = ?$
3. If $8 \times (5 \times 9) = (8 \times n) \times 9$, then $n = ?$
4. If $9(13 - n) = (9 \times 13) - (9 \times 15)$, then $n = ?$
5. If $n \times (9 - x) = (8 \times 9) - (8 \times 7)$, then $n = ?$ and $x = ?$

C. If x stands for a rational number, tell in your own words what each of the following expressions means.

1. $2x - 8$
2. $5x - \frac{1}{3}$
3. $x + 45$
4. $\frac{2x}{3} + 9$
5. $9 - 8x$
6. $14 + \frac{x}{2}$
7. $x - 5$
8. $x + 9$
9. $3x - 8$
10. $3x^2$
11. $\frac{x}{2} + 5$
12. $3x + \frac{x}{2}$
13. $\frac{4x + 5}{2}$
14. $5x^2$
15. $12 - \frac{3x}{4}$
16. $6(x + 5)$
17. $\frac{2x}{3} - \frac{x}{2}$
18. $5x^2 - 9$

D. Write the algebraic expression for each of the following phrases. Let N represent the number.

1. The number increased by 9
2. 5 times the number
3. The number decreased by 7
4. The square of the number
5. Three less than twice the number
6. Five more than the number
7. Five more than $\frac{1}{5}$ of the number

Part Two

A. Solve and check these equations. Indicate what procedure you are using at each step.

1. $x + 14 = 27$
2. $\frac{a}{2} + 3 = 13$
3. $\frac{x}{12} = \frac{1}{3}$
4. $2r + 5 = 9$
5. $\frac{x}{6} = \frac{2}{3}$
6. $\frac{2x}{3} + 14 = 20$
7. $3x = 16 - x$
8. $4x + 3 = 65$
9. $5x - 7 = 2x - 1$
10. $3x + 4 = 7 + 2x$
11. $x + 3 = 8$
12. $x - 8 = 2$
13. $4 + x = 17$
14. $a - 5 = 7$
15. $2n = 8$
16. $\frac{1}{8}q = 15$
17. $2x + 4 = 16$
18. $\frac{n}{3} + \frac{1}{2} = \frac{5}{6}$
19. $5s + 3 = 2s + 15$

B. Remove the parentheses and solve the following equations for the variable.

1. $6(n + 3) = 5(n + 7)$
2. $2(3 + n) + 3(n + 1) = n + 19$
3. $5(x + 1) + 2(2x + 3) = 2(4x + 12)$
4. $7x + 11 = 3(x + 7)$
5. $\frac{1}{2}(2x + 6) = 17$
6. $2(5 + y) + 3y = 2(y + 9) + 1$
7. $0.5(4y + 6) = 0.2(5y + 20)$
8. $\frac{1}{2}(2x + 8) + \frac{1}{3}(3x + 12) = \frac{1}{5}(5x + 20)$
9. $2(2x + 5) + 5(x + 2) = 7(x + 11)$
10. $5(6 + 2x) + 2(x + 3) = 8(x + 7)$
11. $7(x + 6) = 3(x + 40)$
12. $\frac{1}{4}(8x + 32) = \frac{1}{2}(2x + 20)$

C. The following are some common formulas used in mathematics. Solve each problem as directed.

1. The formula for the area of a triangle is $A = \frac{1}{2}bh$. Solve the formula for the base (b) and find the measure of the base of a triangle if the height is 8 inches and the area is 30 square inches.
2. The volume of a rectangular solid can be found by using the formula $V = lwh$. Solve the formula for the length (l), and find l if the volume is 840 cubic inches, the width is 10 inches, and the height is 7 inches.
3. The perimeter, p, of a triangle can be found by using the formula $p = a + b + c$. Solve the formula for c, and find the measure of side c if the other 2 sides measure 12″ and 14″ and the perimeter is 48″.

Part Three

1. A lot is four times as long as it is wide. The distance around the lot is 500 feet. What are the length and the width of the lot?
2. Four times a number decreased by 15 gives the same result as when the number is increased by 3. What is the number?
3. Mark and Andy bought a motor scooter for $84.00. Mark paid twice as much as Andy. How much did each pay?
4. Each of the sides of equal measure of an isosceles triangle measures five inches longer than the base. The perimeter is 64 inches. How long is each side?
5. A twelve-foot board is to be sawed into two pieces so that one piece is 3 feet longer than the other. What will be the length of each piece?
6. The sum of two numbers is 168. One number is three times the other. What are the two numbers?
7. There are 40 more girls than boys in the Sunnyvale High School. The total enrollment is 460. How many girls and how many boys are enrolled in the high school?
8. Martin has a 48-inch steel bar that he wishes to cut in two so that one piece will be five times as long as the other. How long will each piece be?
9. If a certain number is multiplied by 4, the result is the same as if the number is increased by 15. What is the number?
10. A field is twice as long as it is wide. Its perimeter is 360 rods. What are its length and width?
11. Five times a certain number decreased by 18 is equal to twice the number. What is the number?
12. In a certain factory a machinist and his helper together earn $280 a week. The helper earns $\frac{3}{5}$ as much as the machinist. How much does each earn?
13. Mary and Hazel together sold 115 subscriptions to a magazine last week. If Mary had sold 5 more she would have sold twice as many subscriptions as Hazel. How many subscriptions did each girl sell?
14. Each of the equal sides of an isosceles triangle measures $1\frac{1}{2}$ times as long as the base. The perimeter of the triangle is 16 inches. How long is each side?
15. If a certain number is divided by 3, the result is the same as if the number were decreased by 8. What is the number?
16. If one-half a number is decreased by 10, the result is the same as if twice the number were divided by 6. What is the number?

CHAPTER EIGHT

MEASUREMENT

WORDS TO WATCH FOR

approximate	*linear measure*	*precision*
caliper	*liter*	*relative error*
Celsius	*meter*	*significant digits*
error of measurement	*metric system*	*standard unit*
Fahrenheit	*micrometer*	*tolerance*

Measurement has been an essential activity of mankind since building, production of goods, commerce, and travel became important in our society. As our culture has become increasingly technical there has been increasing need for precision, for refined instruments, and for new techniques in measurement.

Measurement is the process of associating number with some property of an object. The number "one" is assigned to some convenient unit of the property that is called the *standard unit,* for example, 1 inch, 1 pound, 1 peck, etc. The process of measurement is that of comparing whatever is being measured to the standard unit.

The uniformity and reliability in standards of measurement that we take for granted today, and that are so essential to every branch of industry and science, are an accomplishment of the present century. From the time of the Egyptian and Roman civilizations until recently, for example, standard units for measurement of length were based upon the human body because this, at the time, was the easiest way of dealing with length. The size of such units, of course, differed from one worker to another.

The inch was legally standardized as the length of 3 barleycorns in the 13th century, but even as late as the American Revolution each of the colonies had its own standards. The length of a yard for measuring

cloth differed from one European country to another, and from one colony to another.

The following facts illustrate several important properties about measurement:

a. Units of measurement are established by the groups using them, and may still differ considerably from one country to another. Most of those that are legally defined and established to make up a system of measures have been established by tradition, rather than from special usefulness. The development of the *metric system,* to be examined later, is an exception to this rule.

b. The unit of measure must be of the same type of measure as the thing being measured. The standard unit for measurement of length is a unit of length; the standard unit for measurement of area is a unit of area, etc.

c. A third characteristic, is that all measurement is *approximate.* Whole numbers were devised to describe the number of elements in a finite set. Correct counting of the set always results in the same number. There are, however, no natural divisions or elements to be counted in a line segment, or the weight of an object. In measurement we are not dealing with the number of objects in a finite set, but with a property of an object. By using increasingly smaller units, and more refined techniques, we can increase the precision of measurement, but there is always a still smaller unit that would provide a more precise measurement. Any measurement is, therefore, approximate.

1. Name a standard unit of measure for each of the following:

a. length
b. weight
c. volume of liquids
d. temperature
e. amount of electricity used
f. amount of gasoline used

2. For which of the measurements above is there a system of units?

3. What instruments are used in the home to make the measurements in Exercise 1?

4. What measurements are important in each of these activities?

a. laying a cement sidewalk
b. catching a train
c. baking a cake
d. assessing a penalty in football
e. doing your homework
f. frying a chop

5. In the cookbook of a generation ago it was common to find directions to use a "pinch of salt," "butter the size of an egg," etc. What system of measures has replaced these units?

6. Different activities call for different kinds of measurement. What units of measure and what measuring instruments are common in each of these situations?

   a. the machine shop
   b. the science laboratory
   c. building a house
   d. flying an airplane

7. The precision of measurement should be adjusted to the purposes for which the measurement is made. A mechanic fitting a piston to the engine of an automobile must measure in units $\frac{1}{6}$ the thickness of a human hair. In what units does a clerk measure in cutting cloth (yard goods) for a customer?

8. In the following pairs of activities, which requires more precision?

   a. baking a cake; making a medical prescription
   b. building a house; overhauling an engine
   c. launching a satellite; catching a train
   d. hanging a door; nailing roof sheathing

9. In what activity would you expect time to be measured?

   a. to the nearest minute
   b. to the nearest second
   c. to the nearest tenth of a second
   d. to the nearest hundredth of a second

QUESTIONS FOR RESEARCH AND DISCUSSION

1. Units of measure that have been in use a long time are frequently derived from natural sources. What is the origin of?

   a. the year
   b. the month
   c. the day

2. Important new activities frequently call for new units and instruments of measurement. Name some of these that developed in regard to the airplane.

3. Many standard units of length are not commonly used today. Some examples are included in the following questions:

   a. How many miles is a "league"?
   b. How many rods is a "furlong"?
   c. How many feet is an "ell"?
   d. How long is a "cubit"?

4. Can you name any units of measurement of weight that are no longer in common use?

5. We think of the present as an age of precision. Yet we still make use of non-standardized units such as "a half-hour's drive." Name other examples of such non-standardized units in common use.

Six segments, a–f, are illustrated below. Copy the table below, and complete it according to the following directions:

1. Without using a ruler estimate the length of each of the six segments, and write your estimate in the first column.
2. With a ruler measure each segment to the nearest $\frac{1}{8}$ inch. Write its length in the second column.
3. Find the difference between your estimate and the length obtained by measuring. Write the difference in the third column.
4. Find the per cent of difference between your estimate and the length.

   HINT: Divide the difference by the length.

a *d*

b *e*

c *f*

Segment	*Estimated Length*	*Length*	*Difference*	*Per Cent Difference*
a	?	?	?	?
b	?	?	?	?
c	?	?	?	?
d	?	?	?	?
e	?	?	?	?
f	?	?	?	?

5. Estimate the length of this page. Then measure to the nearest $\frac{1}{8}$ inch, and calculate the per cent of difference between your estimate and the length.
6. Estimate the length of your pencil to the nearest $\frac{1}{8}$ inch. Then measure it and calculate the per cent of difference between your estimate and the length.
7. Measure each of these segments, and record their lengths, first to the nearest $\frac{1}{4}''$, then to the nearest $\frac{1}{8}''$, then to the nearest $\frac{1}{16}''$.

   a.

   b.

   c.

   d.

When you count money, or the number of pupils in the room, or the elements in a finite set, your answer is an *exact number*. Any correct recount will give you the same number. Numbers from measurements, as we have previously mentioned, are not exact but *approximate*. As has already been stated, you can always arrive at a more precise measurement by using smaller units and more precise instruments. The precision of measurement and the instruments used depend on the *purpose* of the measurement. Coal is weighed to the nearest ton, while the druggist weighs in units of a grain or a fraction of a gram.

State which of the following statements, 1 through 7, use exact numbers and which use approximate numbers:

1. I have $1.27 in my purse.

2. The average height of the players on the basketball team is 6′1″.

3. There are 1858 pupils enrolled in Washington High School this year.

4. The lowest temperature last winter was $^{-}5.3$ degrees Fahrenheit.

5. Mr. Jacobsen's income last year was $6725.93.

6. The circumference of the earth at the equator is 25,000 miles.

7. The fat content of the milk from Lakeside Dairy is 4.1%.

8. Eric is working from a blueprint that reads "tolerance $\pm\frac{1}{64}$ inch." His shop teacher told him that tolerance is the maximum error allowed for a given measurement. That is, if the blueprint specified a length of $1\frac{3}{4}''$ for a particular object, then the object could be as small as $1\frac{3}{4} - \frac{1}{64}$, or $1\frac{47}{64}''$, in length, or as large as $1\frac{3}{4} + \frac{1}{64}$, or $1\frac{49}{64}''$, in length. Thus the object, to be acceptable, must be greater than or equal to $1\frac{47}{64}''$ but less than or equal to $1\frac{49}{64}''$. If Eric finds a measurement given as $1\frac{1}{2}''$, what is the maximum allowable size? Use the same tolerance as above.

9. What is the smallest allowable size if the given measurement is $1\frac{1}{2}$ inches? Use the same tolerance as in Exercise 8.

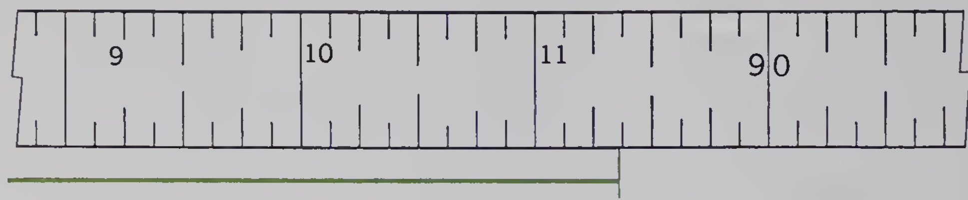
10. If the tolerance is $\pm\frac{1}{32}''$, what is the maximum allowable length for a part whose dimension is given as $5\frac{5}{16}''$?

11. The measure of a diameter of a piston is given on a blueprint as $3\frac{1}{4}''$, with a tolerance of $\pm\frac{1}{64}''$. Which of these dimensions would be acceptable?

a. $3\frac{15}{64}''$ **b.** $3\frac{3}{32}''$ **c.** $3\frac{1}{2}''$ **d.** $3\frac{1}{32}''$ **e.** $3\frac{17}{64}''$

12. Mathematicians follow a rule that "a measurement stated with no indication of how it was obtained is assumed correct to the smallest unit shown." What is the difference between the measurements 7 lb. and 7.0 lb.?

The regulation distance from home plate to first base on a baseball diamond is 90 feet. In checking the distance on the school diamond last spring Mike and Joe found the distance, to the nearest inch, to be 89 feet 11 inches.



First base is a bag fastened by a strap to a peg in the ground so that the base is slightly movable. The measurement varies with the position of the bag. Therefore a more precise measurement than what was found by Joe and Mike is unnecessary. The *error of measurement* was about an inch, which, under the circumstances, was permissible. Used in this sense, error of measurement does not imply a mistake. It indicates the precision of measurement.

1. Remember! The precision of a measurement should be in accord with the purpose for which the measure is being made. In which of the following measurements would the least error be permissible? In which would the largest error be permitted?

a. Measuring the surface of a barn to determine the amount of paint to buy

b. Measuring the boards needed to build a desk

c. Measuring the parts of a watch

d. Measuring the cloth in making a dress

2. The unit of measurement is adapted to the required precision. In Figure 1 below, the measurement is reported to the nearest foot, so the unit of measurement is a foot. What is the measurement in Figure 1?

Figure 1

3. What unit of measure is being used if the measurement in Figure 1 is reported as 66 feet 11 inches?

The airline distance from New Orleans to Denver measures 1,000 miles, to the nearest ten miles. A certain city lot measures 60 feet, to the nearest foot. A diameter of a watch crystal, to the nearest $\frac{1}{4}$ inch, measures $1\frac{1}{4}$ inches. Since the error of measurement, unless otherwise specified, is one-half of the smallest unit reported, we can state the error of each measurement. The first ($\frac{1}{2}$ of 10 miles) is the largest. However, the third ($\frac{1}{2}$ of $\frac{1}{4}$-inch) is probably the most important, since a watch crystal with such a large error of measurement probably would not fit. The importance of the error can usually be determined by comparing it to the measurement itself. This comparison is called the *relative error of measurement.* As a formula this can be stated as:

$$\text{relative error of measurement} = \frac{\text{error}}{\text{measurement}}$$

Thus the relative error of measurement, when the distance from New Orleans to Denver is reported as 1000 miles to the nearest 10 miles, is $\frac{5}{1000}$, or 0.005, or 0.5%. In the following problems, express the relative error in per cent.

1. What is the relative error of measurement in the width of a city lot if the measure, reported to the nearest foot, is 60 feet?
2. What is the relative error of measurement of a $1\frac{1}{4}''$ watch crystal, if the measure is reported to the nearest $\frac{1}{4}$-inch?
3. A weight is reported as 83.5 lb. The error of measurement is 0.05 lb. ($\frac{1}{2}$ of 0.1 lb.). What is the relative error in per cent?
4. The relative error of measurement determines the "degree of precision" of the measurement. The *smaller* the relative error the *greater* is the precision. By calculating the relative errors of measurement, determine which of these measurements has the greatest degree of precision and which has the least.

   a. 6 ft. **c.** 8 in. **e.** 0.01 in.
   b. 6.0 ft. **d.** 8.0 in. **f.** 0.001 in.

5. The length of a bridle path through a park is reported as 120 feet, and the width as 9 feet. Which of these measurements has the greater degree of precision?
6. A measurement is reported as $2'\ 1\frac{3}{16}''$. The error of measurement is $\frac{1}{32}''$. Why? What is the relative error of measurement expressed as a per cent?
7. Jim ran 100 yards in 10.3 seconds in a recent track meet. What was the error of measurement?
8. What is the relative error of measurement, expressed as per cent?

9. Find the relative error of measurement as a per cent for each of the following measurements:

a. $16' 7\frac{1}{2}''$
b. 9.8 sec.
c. 4 min. 12 sec.
d. 8 lb. 6 oz.
e. 9 hr. 25 min.
f. 18 miles 525 ft.
g. 10.0 sec.
h. 4 min. 0 sec.
i. 5 lb. 0 oz.
j. 6 yd. 9 in.

10. If the measure of a diameter of a bearing being constructed in the machine shop is specified as $2.5'' \pm .005''$, what per cent of error is permissible?

11. Express as per cent the relative error for each of the following measurements:

a. $9\frac{1}{2}$ lb. b. $9\frac{2}{4}$ lb. c. $9\frac{4}{8}$ lb. d. $9\frac{8}{16}$ lb.

12. Express as per cent the relative error for each of the following measurements:

a. 4 sec. b. 0.4 sec. c. 0.04 sec. d. 0.004 sec.

13. The terminology used in expressing ideas on precision of measurement may be summarized in the outline which follows:

Measurement	*Unit of Measurement*	*Meaning*	*Error*	*Relative Error*
56 lb.	1 lb.	Nearer to 56 lb. than to 55 lb. or 57 lb.	$\frac{1}{2}$ lb.	$\frac{0.5}{56} = 0.9\%$
28.5 ft.	0.1 ft.	Nearer to 28.5 ft. than to 28.4 ft. or to 28.6 ft.	0.05 ft.	$\frac{0.05}{28.5} = 0.18\%$
$13\frac{3}{8}$ in.	$\frac{1}{8}$ in.	Nearer to $13\frac{3}{8}$ in. than to $13\frac{2}{8}$ in. or to $13\frac{4}{8}$ in.	$\frac{1}{16}$ in.	$\frac{\frac{1}{16}}{13\frac{3}{8}} = 0.5\%$

Explain the meaning of a measurement reported as $13\frac{5}{16}$ lb.

14. Explain the meaning of each of these measurements.

a. 8 in. b. $33\frac{1}{3}°$ c. $19\frac{1}{2}$ sq. ft. d. 9.7 mi.

15. What is the unit of measure in each measurement reported in Exercise 14?

16. Express as per cent the relative error for each measurement in Exercise 14.

17. State the meaning of each of these measurements.

a. $15\frac{1}{2}$ in. b. $15\frac{2}{4}$ in. c. $15\frac{4}{8}$ in. d. $15\frac{8}{16}$ in.

18. Which of the measurements reported in Exercise 17 has the greatest error of measurement? Which has the least?

Computation with Approximate Numbers

The measurement of a garden is reported as 5.6 rods long and 4.8 rods wide. What is its area? The computation is:

$$5.6 \times 4.8 = 26.88$$

But is its area 26.88 square rods?

1. The product of two numbers from measurement cannot have a greater degree of precision than the factors. Since the measurements are to the nearest tenth of a rod, the errors of measurement are each 0.05 rod. In regard to the length, this is a relative error of 0.9%. Explain how this is obtained.
2. For the width this is 1%. How was this obtained?
3. In stating the area as 26.88 square rods, the error of measurement is 0.005 square rods. How was this obtained?
4. Then the relative error of measurement if the area is stated as 26.88 is 0.02%. Explain the computation.
5. If we round the area to 26.9 square rods, the relative error becomes 0.2%. Show the computation.
6. Is this relative error greater or less than the relative error of measurement of the length and the width?
7. If we round the area to 27 square rods, the error of measurement is 1.85%. Check to show that this is correct.
8. Thus, in order that the relative error of measurement of the area should not be less than that of the length or the width, the area should be rounded from 26.88 to 27. Explain why it must be 27, rather than 26.9 or 26.88.
9. In general, then:

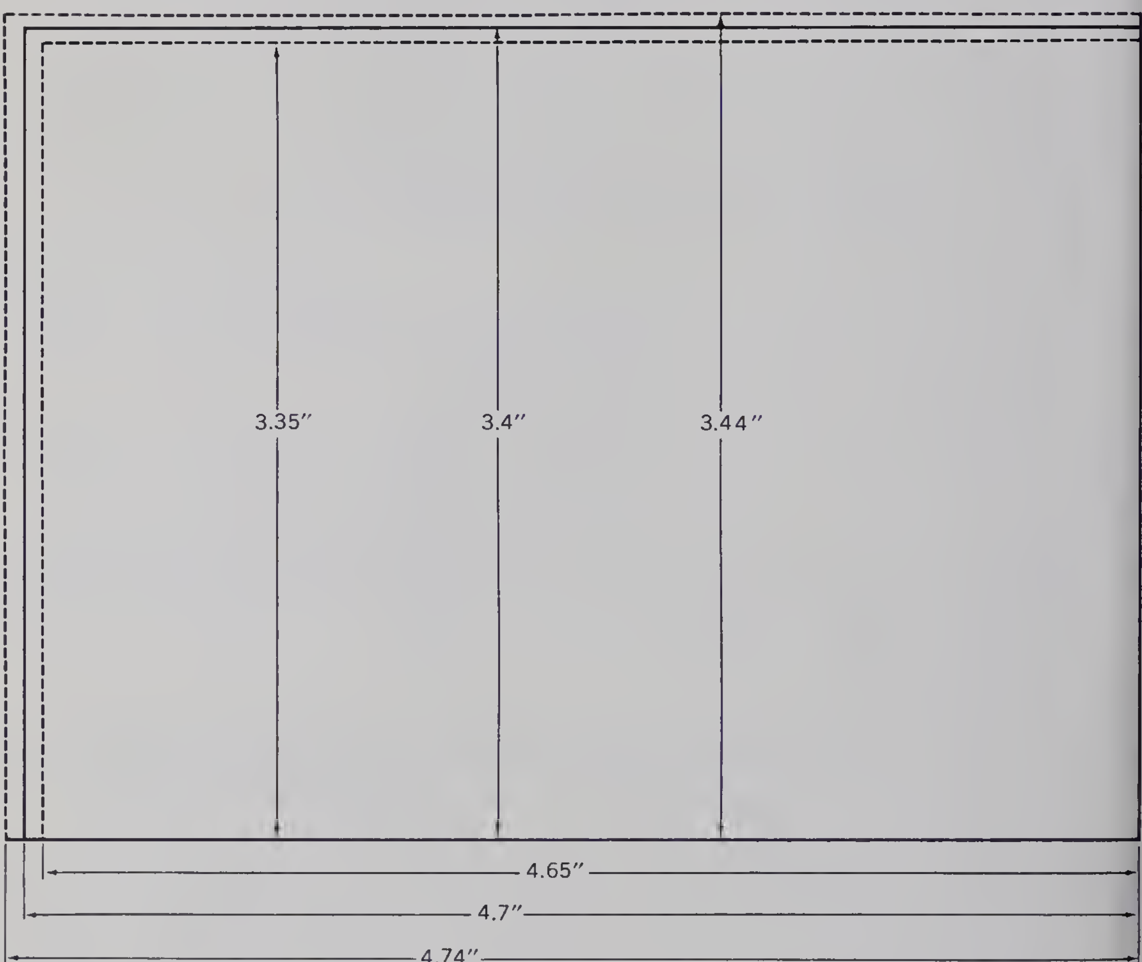
The product of numbers from measurement should be rounded to a degree of precision not to exceed that of the factors.

Use this Principle in stating the areas of rectangles whose dimensions are as follows:

	Length	Width		Length	Width
a.	16.5 rd.	9.7 rd.	**d.**	1.75 in.	0.94 in.
b.	9.2 ft.	6.7 ft.	**e.**	6.25 rd.	4.75 rd.
c.	3.5 yd.	1.7 yd.	**f.**	15.9 in.	9.7 in.

10. The need to reconcile the error of measurement of the product to that of the factors may be illustrated in a diagram. A piece of sheet copper is measured and found to be 4.7 inches long and 3.4 inches wide. Therefore the measuring instrument used is accurate to a tenth of an inch. Since the dimensions are rounded to the nearest tenth of an inch, the actual length could be as little as 4.65 inches, or as great as 4.74 inches. In like manner, the width of 3.4 inches could be as little as 3.35 inches or as great as 3.44 inches. Thus we have maximum and minimum possibilities for the area, as well as the computed area.

Minimum:	$3.35 \times 4.65 = ?$
Computed:	$3.4 \times 4.7 = ?$
Maximum:	$3.44 \times 4.74 = ?$



These three measures are shown in the above Figure. Any area within the maximum and minimum limits is within the limits of the precision of the reported measurements. To take this into account, the reported area should be rounded to 16 square inches, with an error of 0.5 square inches.

11. The error of measurement of the area is shown geometrically by the strip between the maximum and minimum dimensions. Calculate this area and see if it is approximately 0.5 in square inches.

12. Find the relative errors of measurements of 4.7 in., 3.4 in., and 16 square inches to determine if they are stated with approximately the same precision.

13. What is the relative error of the computed area of 15.98 square inches?

14. The term *significant digits* of a numeral applies to all the digits of a numeral except the zeros used as "place holders" to locate the decimal point. Thus 47500 has three significant digits as does 0.00475. However 400500 and 0.004005 each have four significant digits. How many significant digits do 415.0 and 0.004150 have?

15. State the number of significant digits in each of the following numerals.

a. 310	**c.** 618.03	**e.** 0.03150	**g.** 0.007635
b. 0.0573	**d.** 728.0	**f.** 88.003	**h.** 912.050

16. A common rule for adjusting the relative error of the product to that of the reported dimensions can be stated as follows:

Rule: Round the product to the same number of significant digits as are in the factor with the least number of significant digits.

For example:

How many square rods are in a garden that measures 7 rods wide and 12 rods long?

The product of 7 and 12 is 84. However, this must be rounded to a number with one significant digit. Why? The area is then written as 80 square rods. Why? Check the relative errors to see if the precision is appropriate.

17. Use the rule of Exercise 16 to state the areas of these rectangles with the given measurements. Be sure to use the same degree of precision used in the measurements which follow.

Length	*Width*	*Length*	*Width*
a. 6 ft.	4 ft.	**f.** 17.2 rd.	8.1 rd.
b. 12.6 ft.	8.5 ft.	**g.** 12.1 rd.	9.8 rd.
c. 8 ft. 3 in.	6 ft. 4 in.	**h.** 16.2 ft.	8.0 ft.
d. 185 rd.	12.2 rd.	**i.** 9 in.	6 in.
e. 85 rd.	62 rd.	**j.** 18 ft.	7 ft.

Measurement in Science

When scientists and engineers exchange information throughout the world they use *metric* units of measurement. Many large industries, like those that manufacture cameras, films, and projectors, also use metric units.

The *metric system* has the following important advantages that have led to its exclusive use in all countries except Canada and the United States. (Great Britain is changing over to the metric system at present.)

a. A decimal relationship, such as we have in our coinage, exists between units.
b. The prefixes of the units are self-defining: centi- means "one hundredth of"; kilo- means "one thousand."
c. The direct relationship of units of capacity and weight to units of measure: for example, the unit of weight is the gram, which is the weight of 1 cubic centimeter of water at a temperature of 4° Celsius, or 4° *C*. (Later in the chapter we will discuss the "Celsius" scale for measuring temperature.)

The basic unit of the metric system for measuring length is the *meter, m.*, which is about 39.37 inches. The other units in the system are as follows:

1 millimeter (mm.)	= 0.001 meter
1 centimeter (cm.)	= 0.01 meter
1 decimeter (dm.)	= 0.1 meter
1 dekameter (dkm.)	= 10 meters
1 hectometer (hm.)	= 100 meters
1 kilometer (km.)	= 1000 meters

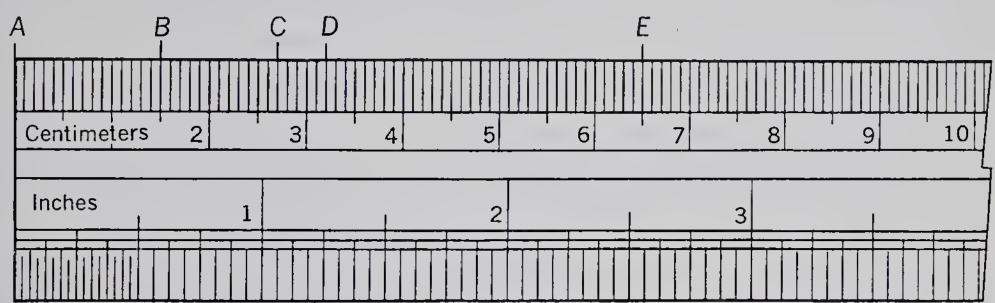
1. One centimeter is equal to how many millimeters?
2. The prefixes centi- and kilo- are defined above. What is the meaning of milli-, deci-, deka-, and hecto-?
3. What is the ratio of a millimeter to a meter?
4. What is the ratio of a kilometer to a meter?
5. How many millimeters does 42 centimeters equal?
6. How many centimeters does 35 millimeters equal?
7. State a rule for expressing millimeters as centimeters. Give illustrations.
8. State a rule for expressing kilometers as meters. Give illustrations.

As you have discovered in previous exercises on measurement, we are making approximations when we measure. The mathematical symbol for "approximate" is $\approx$. Therefore the statement "1 inch $\approx$ 2.5 cm." means "1 inch is approximately 2.5 cm."

Here are some more "approximate" equivalents of the English and metric systems.

1 centimeter $\approx$ 0.4 inch
1 meter $\approx$ 1.09 yards
1 kilometer $\approx$ 0.62 mile (or about $\frac{5}{8}$ mile)

1. The ruler below is divided into centimeters and inches. The upper part represents centimeters, and the bottom part represents inches. Each centimeter is divided into ten equal parts. What is the name of each of these parts?



2. How many millimeters is the distance from A to B? A to C? D to E?
3. Write these approximations.
1 inch $\approx$ _?_ centimeters; 1 foot $\approx$ _?_ centimeters.
4. Jim has a 35 mm. camera. (The "35 mm." is the width of the film that the camera uses.) To the nearest tenth of an inch, about how wide is the film? Make your calculations from the equivalents given above, then check your answer using the ruler above.
5. Two other popular sizes of film are 16 mm. and 8 mm. What is the approximate width of each of these films? Give your answer to the nearest tenth of an inch.
6. International track and field records are frequently stated in metric units. Can you give a reason why?
7. What fraction of a mile is 1500 meters? Write the answer as a decimal rounded to the nearest hundredth.
8. One of the events in the Olympic games is the 100-meter dash. How many yards is 100 meters?

The basic unit of weight in the metric system is the *gram*. (It is the weight of 1 cubic centimeter of water at a temperature of 4° Celsius (or Centigrade), or 4° *C*, which we will examine later.)

Metric Units of Weight		*Commonly Used Equivalents*
10 milligrams (mg.)	= 1 centigram (cg.)	1 gram ≈ 0.035 oz.
10 centigrams	= 1 decigram (dg.)	1 kg. ≈ 2.2 lb.
10 decigrams	= 1 gram (g.)	1 MT ≈ 2204.6 lb.
10 grams	= 1 dekagram (dkg.)	1 oz. ≈ 28.4 g.
10 dekagrams	= 1 hectogram (hg.)	1 lb. ≈ 0.453 kg.
10 hectograms	= 1 kilogram (kg.)	
1000 kilograms	= 1 metric ton (MT)	

1. A golf ball weighs 1.62 oz. Approximately, how many grams does it weigh?

2. Passengers on a transatlantic flight are permitted to check 30 kilograms of baggage. Approximately how many pounds is this?

3. A shipload of wheat weighed 6000 metric tons. Approximately how many English tons does it weigh?

4. Jim weighs 130 pounds. Approximately how many kilograms does Jim weigh?

5. Metric units of weight are widely used in science, medicine, and pharmacy. A vitamin tablet is advertised as follows: "Each 30 grams contain 15 g. protein; 4.5 mg. vitamin B_1; 30 mg. niacin." Approximate these weights in ounces.

6. The ratio between each unit and the next larger unit in the metric system is always 1 to 10. In the English system the ratios vary. In units of weight, for example,

Units	*Ratios*
Ounce to pound	1 to 16
Pound to hundredweight	1 to 100
Pound to ton	?

Complete the table. Then, set up a similar table to show the ratios between each unit and the next in each of the following problems.

7. The United States units of money: mill, cent, dime, dollar.

8. The English units of length: inch, foot, yard, rod, mile.

9. The units of liquid measure: teaspoon, tablespoon, cup, pint, quart, gallon.

Note: Refer to the Tables in the back of the book if you have forgotten any of the measurements.

In most countries other than Canada and the United States, the unit of measure for capacity is the liter (pronounced "lee-ter"). A liter is 1000 cubic centimeters. To further illustrate the direct relationship between the metric units of measure, a liter of water weighs 1 kilogram at 4° C. The metric units of capacity have the same prefixes as the units for measuring length.

Metric Units

10 milliliters	=	1 centiliter (cl.)
10 centiliters	=	1 deciliter (dl.)
10 deciliters	=	1 liter (l.)
10 liters	=	1 dekaliter (dkl.)
10 dekaliters	=	1 hectoliter (hl.)
10 hectoliters	=	1 kiloliter (kl.)

English Equivalents

Liquid

1 liter	≈	1.056 quarts
1 quart	≈	0.95 liters
1 gallon	≈	3.8 liters

Dry

1 liter	≈	0.91 quarts
1 quart	≈	1.1 liters
1 hectoliter	≈	2.8 bushels

Remember! In changing metric measures into English measures or English measures into metric measures, your answers are *approximate.*

1. A hectoliter is how many gallons?

2. How many liters are in 1 peck?

3. A drum of oil contains 50 gallons. How many liters does it contain?

4. How many 50-gallon drums are in a kiloliter?

5. If gasoline sells at 23¢ a liter in France, this is how much per gallon? Round your answer to the nearest cent.

6. A cubic storage tank measures 1 meter on each edge. How many liters will it contain?

7. If the storage tank (Exercise 6) were filled with gasoline worth 20¢ a liter, how much would the gasoline be worth?

8. If a car travels 8 miles on a liter of gasoline, how many miles per gallon does it travel? Round your answer to the nearest mile.

9. A storage tank holds 200 hectoliters of gasoline. How much is it worth at 80¢ per gallon?
HINT: Change 200 hectoliters to gallons. Round this answer to the nearest whole gallon.

10. The average car in this country uses approximately 600 gallons of gasoline per year. How many liters is this?

11. A car travels 15 miles per gallon of gasoline. How many liters (rounded to the nearest liter) would it use in traveling 120 miles?

Table of Equivalents		
1 kg.	$\approx$	2.2 lb.
1 lb.	$\approx$	0.45 kg.
1 m.	$\approx$	1.1 yd.
1 yd.	$\approx$	0.9 m.
1 km.	$\approx$	0.62 mi.
1 mi.	$\approx$	1.6 km.

When athletes representing the nations of the world come together every four years for the Olympic games, the metric system instead of the English system is used for measuring distances. The unit of measure is the meter, which is approximately equal to 39.37 in. Instead of the 100-yard dash, they have the 100-meter dash (about 109 yards). Instead of the mile run (1760 yards) they have the 1500-meter run (about 1640 yards). Metric units are used for the shot-put, vaulting, jumping, and swimming contests. Remember that the conversions are only approximate.

1. The Olympic record for the 1500-meter swim is 17 minutes 1.7 seconds. What fraction of a mile is 1500 meters?

2. A certain boxer in the Olympics weighed 80 kg. Would he be a light-heavyweight (160 lb.–175 lb.) or a heavyweight (over 175 lb.)?

3. The world's record for the 1-mile run in 1964 was 3 min. 54.1 sec. The world's record for the 1500-meter run was 3 min. 35.6 sec. In which race did the runner travel faster?

4. In 1965 the world's record for the 100-yd. dash was 9.1 sec. The Olympic record for the 100-meter race was 10.0 sec. In which race did the runner travel faster?

5. The Olympic record for the javelin throw is 85.7 meters. This is about how many feet?

6. For many years the Olympic record for the 3500-meter walk has been 14 minutes 55 seconds. This is at the rate of how many miles per hour?

7. A pole vaulter at the 1964 Olympics reached a height of approximately 5.1 m. Is this more or less than 17 feet?

8. The record for the 50,000-meter walk set in 1964 was 4 hours 11 minutes 12.4 seconds. This is approximately how many miles per hour?

9. At one time the women's record for the discus throw was 57.05 meters. What was the women's record for the discus throw? Express your answer in feet and inches.

10. In the Olympic games, the athletes run 100 meters instead of 100 yards. How much longer than 100 yards is this race?

SPECIAL REPORT

Make a report on the Olympic games, including a brief history, the purposes of the games, and a description of the most recent Olympic games.

In countries where the metric system is in common use, temperature is measured on the *Celsius* (or Centigrade) scale. In examining the Celsius scale, it will become apparent why those who use the metric system also use the Celsius scale. This is also commonly true in scientific work.

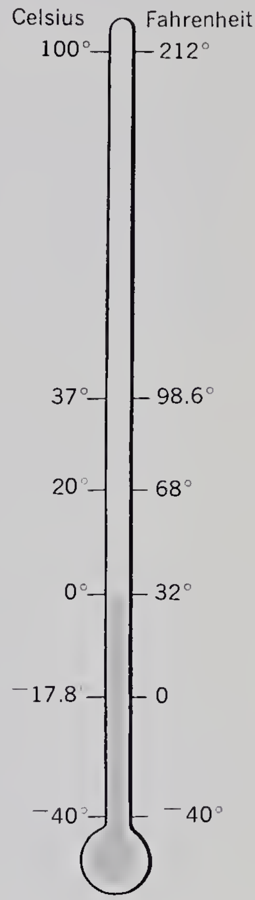
1. On the thermometer used in this country, temperature is measured in degrees *Fahrenheit, F*. (See the Figure at the right.) The freezing point of water is 32° F and the boiling point is 212° F. How many degrees are there between the boiling and the freezing points of water?

2. In Europe where the metric system is used, temperature is measured in degrees Celsius, C. (See the Figure at the right.) What is the freezing point of water on this scale? The space between the boiling and freezing points represents how many degrees?

3. The average room temperature for public buildings is 68° on the Fahrenheit scale. What is this on the Celsius scale?

4. What is the normal body temperature on each scale?

5. What temperature reading is the same on both scales?



Celsius and Fahrenheit Scales

6. The Rule for finding the Celsius temperature when the Fahrenheit is given, is as follows:

Subtract 32 from the Fahrenheit temperature, and multiply the result by $\frac{5}{9}$.

Use the Rule above to change 50° Fahrenheit to Celsius.

7. The Formula for finding the Fahrenheit temperature reading when the Celsius is given, is: $F = \frac{9}{5}C + 32$

Rewrite the Formula above in order that you can find the Celsius temperature reading when the Fahrenheit is given. Compare the formula you obtained with the Rule stated in Exercise 6 above.

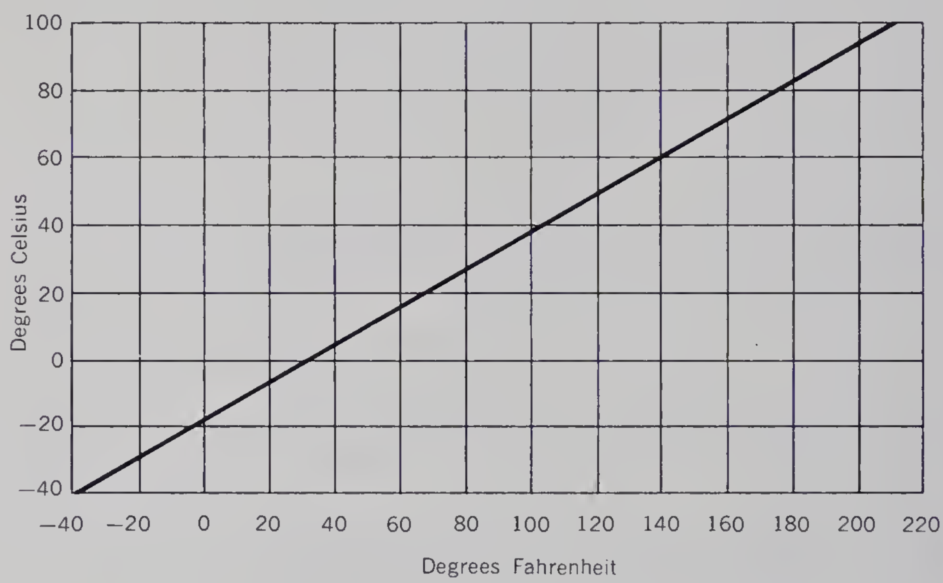
8. Change the following to Fahrenheit degrees by using the Formula.

a. 45° C	**d.** 5° C	**g.** 100° C
b. 24° C	**e.** 80° C	**h.** ⁻273° C
c. 22° C	**f.** ⁻40° C	**i.** 0° C

9. Change the following to Celsius degrees.

a. 82° F	**d.** 20° F	**g.** 98.6° F
b. 103° F	**e.** 15° F	**h.** 212° F
c. 10° F	**f.** ⁻40° F	**i.** 0° F

10. The conversion from Fahrenheit to Celsius, and vice versa, is simplified by the use of a graph, as in the Figure below. Select a given Celsius scale temperature on the left (as 40°) and follow across to the diagonal line. The reading directly below on the Fahrenheit scale, 104°, is the equivalent Fahrenheit temperature. Use the Formula to see if this is correct.



11. The highest and lowest temperatures in several cities, recorded one day in March, were as follows:

	High	*Low*		*High*	*Low*
Chicago	43	34	Las Vegas	90	49
El Paso	76	38	Phoenix	85	46
Helena	23	⁻5	Spokane	25	⁻7

The difference between the high and low temperature is called the *range* in temperature. What is the range for each of the cities?

12. Use the Graph above to express the high and low temperatures for each city in degrees Celsius. Do the same for the range.

A. *Add:*

1. $6\frac{1}{4} + 2\frac{3}{8}$ **2.** $5\frac{1}{2} + 3\frac{7}{12}$ **3.** $1\frac{1}{5} + 3\frac{2}{3}$ **4.** $17\frac{2}{3} + 15\frac{1}{4}$

5. $\begin{array}{r} 2\frac{1}{6} \\ 5\frac{2}{3} \\ \underline{3\frac{1}{4}} \end{array}$ **6.** $\begin{array}{r} 27\frac{7}{10} \\ 35\frac{3}{4} \\ 9\frac{1}{2} \\ \underline{21\frac{3}{5}} \end{array}$ **7.** $\begin{array}{r} 2\frac{3}{8} \\ 5\frac{5}{16} \\ 4\frac{1}{4} \\ \underline{8\frac{1}{2}} \end{array}$ **8.** $\begin{array}{r} 76\frac{2}{3} \\ 13\frac{5}{6} \\ 8\frac{11}{12} \\ \underline{24\frac{3}{4}} \end{array}$ **9.** $\begin{array}{r} 19\frac{1}{6} \\ 39\frac{3}{8} \\ 6\frac{7}{10} \\ \underline{12\frac{1}{2}} \end{array}$

B. *Subtract:*

1. $\begin{array}{r} 5\frac{7}{8} \\ \underline{3\frac{3}{4}} \end{array}$ **6.** $\begin{array}{r} 27\frac{2}{5} \\ \underline{19\frac{1}{4}} \end{array}$

2. $\begin{array}{r} 9\frac{11}{12} \\ \underline{4\frac{3}{8}} \end{array}$ **7.** $\begin{array}{r} 63 \\ \underline{17\frac{3}{7}} \end{array}$

3. $\begin{array}{r} 16\frac{3}{5} \\ \underline{8\frac{3}{10}} \end{array}$ **8.** $\begin{array}{r} 29 \\ \underline{15\frac{2}{5}} \end{array}$

4. $\begin{array}{r} 27\frac{2}{3} \\ \underline{13\frac{1}{4}} \end{array}$ **9.** $\begin{array}{r} 76\frac{1}{2} \\ \underline{59\frac{2}{3}} \end{array}$

5. $\begin{array}{r} 17\frac{1}{5} \\ \underline{9\frac{3}{4}} \end{array}$ **10.** $\begin{array}{r} 95\frac{1}{5} \\ \underline{39\frac{1}{4}} \end{array}$

C. *Multiply:*

1. $16 \times 4\frac{3}{4}$ **3.** $1\frac{4}{5} \times 4\frac{3}{7}$ **5.** $16\frac{2}{5} \times 4\frac{5}{6}$ **7.** $2\frac{1}{6} \times 7\frac{9}{10}$

2. $24 \times 1\frac{3}{8}$ **4.** $8\frac{1}{3} \times 9\frac{5}{12}$ **6.** $3\frac{8}{9} \times 2\frac{3}{5}$ **8.** $9 \times 6\frac{2}{3}$

D. *Divide:*

1. $2\frac{9}{10} \div 1\frac{3}{4}$ **3.** $\frac{3}{8} \div 15$ **5.** $2\frac{13}{14} \div 6\frac{1}{2}$ **7.** $15\frac{1}{3} \div 4\frac{7}{12}$

2. $12 \div \frac{8}{9}$ **4.** $4\frac{11}{12} \div 6\frac{1}{2}$ **6.** $4\frac{1}{5} \div 2\frac{7}{8}$ **8.** $8\frac{2}{5} \div 3\frac{7}{10}$

E. *Add:*

1. ${}^{+}6 + ({}^{+}8)$ **3.** ${}^{-}16 + ({}^{-}17)$ **5.** ${}^{+}21 + ({}^{-}26)$

2. ${}^{+}18 + ({}^{-}9)$ **4.** ${}^{-}19 + ({}^{+}20)$ **6.** ${}^{-}17 + ({}^{+}21)$

F. *Subtract:*

1. ${}^{+}14 - ({}^{+}7)$ **3.** ${}^{-}15 - ({}^{+}19)$ **5.** ${}^{+}13 - ({}^{+}19)$

2. ${}^{+}8 - ({}^{+}11)$ **4.** ${}^{-}18 - ({}^{+}17)$ **6.** ${}^{-}12 - ({}^{+}6)$

If you need more practice, turn to the Practice Exercises on page 465 and following. If not, you may work in the Experts' Corner on the following page.

How Long Is an Inch?

In the fourteenth century, King Edward II proclaimed that the English inch should be the length of "three barley-corns, dry, round, and laid end-to-end." While he thought he had settled the basic question of measuring length, there has been argument about the length of the inch ever since. Until recently, the British inch was 2.53999560 cm.; the Canadian inch was 2.54000000 cm.; and the United States inch was 2.54000508 cm. Even these small differences created problems in fine precision manufacturing. During World War II, when other countries needed our gauge blocks (the precise, polished-steel measuring standards needed for making planes and guns), each had to be custom made. War supplies, as a result, were sometimes delayed for months.

Scientists and engineers have urged governments to get together to straighten out this small difference, but the expense of changing over has prevented it. American map-makers, for example, point out that it would throw some of their maps several feet out of scale. Recently, however, the United States, Canada, New Zealand, Australia, and South Africa agreed to adopt the Canadian size inch.

Study the paragraphs above and answer the following questions assuming that the adoption is not yet made.

1. We speak of an "English system" of linear measurement. Is there one system or are there three systems? Explain.
2. How would the British and United States foot rules compare with the Canadian foot rule?
3. The U.S. inch is how much longer than the British inch?
4. How much difference would this make on a foot rule?
5. Do you think you could tell the difference by comparing a British and United States foot rule? Explain your answer.
6. Explain what the map-makers mean when they say a change would throw their maps "several feet out of scale."
7. From the statement above it appears that the three systems of linear measurement are actually based on the metric system. Explain.
8. Why is it not simpler to change to the metric system, rather than to change the size of the British and U.S. inch?
9. In 1965 the British Government announced plans for the conversion to the metric system over a ten-year period to promote international trade. Give illustrations of industries that will have heavy expenses in converting units of weight, length, and volume.

Mr. Brown and his family decided to take a motor trip to Mexico. Mr. Brown had been told that in Mexico the metric system of measurement is used, so the family decided to learn the differences between the English system and the metric system. Round all answers in the following problems to the nearest tenth. Refer to the tables on the previous pages as needed.

1. Mr. Brown's son, John, learned that the standard unit of linear measure in Mexico is the meter. Is this longer or shorter than the yard? How many inches longer or shorter?

2. A kilometer is 1000 meters. How many feet is a kilometer?

3. One mile is approximately 1.61 kilometers. If the distance to the next town is 60 miles, how many kilometers would it be?

4. If one kilometer is approximately 0.62 miles, how many miles is it to the next town if the sign says "95 km."?

5. The first road sign that Mr. Brown and his family came to is pictured at the right. Translated, the sign reads "Maximum speed limit is 100 kilometers per hour." How many miles per hour is this?

VELOCIDAD
MAXIMA
100
KM. POR H.

6. If the sign says that the speed limit is 50 kilometers per hour, how many miles per hour is this?
HINT: Use your answer from Exercise 5.

7. The unit of liquid measure in the metric system is the liter, which is a little more than one quart. (See page 307.) We find that 1 gallon is approximately 3.8 liters. How many liters would there be in 10 gallons?

8. Mr. Brown said that his gasoline tank holds 20 gallons. How many liters will the tank hold?

9. If Mr. Brown gets 18 miles to a gallon of gasoline, how many miles per liter can he get?

10. Farther down the road they came to a service station selling gasoline for 10¢ a liter. How much is the cost of a gallon?

11. According to the road map the distance between two cities was 72 km. It took one hour to make the trip. How many miles per hour did they average?

12. On their return trip they left Guaymas at 1:00 P.M. and planned to arrive in Nogales, a distance of 432 km., at 7:00 P.M. How many miles per hour must they average to arrive on time?

13. The Browns traveled 1568 miles while in Mexico. This is how many kilometers? Give your answer to the nearest kilometer.

14. They had purchased 426 liters of gasoline while in Mexico. To the nearest mile, how many miles per gallon of gasoline did they average?

15. The kilogram is the standard unit of weight in Mexico. One pound is approximately 0.453 kilograms. (See page 306.) Is a kilogram greater or less than a pound?

16. The tire pressure in Mr. Brown's tires is 28 pounds per square inch. The pressure is how many kilograms per square inch?

17. John weighed himself on a scale outside a store, and a card said he weighed 45 kilograms. Express his weight in pounds.

18. Jimmie Brown saw a sign on a bridge that said "Capacity 2540 kilos (kilograms)." To the nearest ton, this is how many tons?

19. Jimmie drank 800 milliliters of milk for breakfast. What part of a quart is 800 milliliters, if there are 1000 milliliters in a liter? Round your answer to the nearest hundredth.

20. Normally the Browns drink 3 quarts of milk daily. If there are approximately 946.3 milliliters in one quart, how many liters would they normally consume? Round your answer to the nearest tenth.

21. As they traveled southward, they wondered how many acres there were on the ranch they were going to visit. Mrs. Brown said the area of the ranch is 500 square kilometers. If 1 square kilometer is approximately 247 acres, about how many acres does the ranch contain?

22. How many square kilometers are there in a ranch that contains 3000 acres? Round your answer to the nearest square kilometer.

23. How many square kilometers are there in a section? (A section is a mile square.) Round your answer to the nearest tenth.

24. The altitude (the measure of the distance above sea level) of Mexico City is 2241 meters. To the nearest foot, this is how many feet?

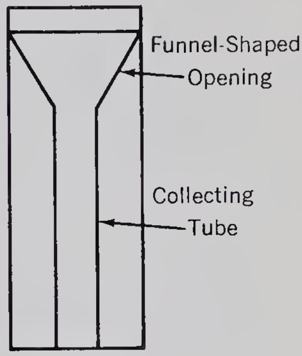
25. A neighboring mountain peak is 15,016 feet high. To the nearest meter, this is how many meters?

26. The Pan American Highway that runs from Alaska to Cape Horn passes through Mexico. It enters Mexico at Juárez, opposite Laredo, Texas, and passes through Mexico City, 1979 kilometers from Juárez. Using the fact that a kilometer is approximately $\frac{5}{8}$ of a mile, find the approximate number of miles it is from Juárez to Mexico City.

27. The last large city on the Pan American Highway in Mexico before entering Guatemala is Tuxtla Gutiérrez, 1088 kilometers below Mexico City. Using the approximation given in Exercise 26, find the approximate number of miles it is from Juárez to Tuxtla Gutiérrez.

The weather has always been one of man's greatest concerns. "Talking about the weather" is a typical pastime, thus revealing our interest in it. Although we are not yet able to do much about the weather, we have learned a lot about it and are able to predict weather changes with ever-increasing accuracy.

1. The amount of rainfall is measured by a *rain-gauge.* (The Figure at the right shows the inside.) Those used at weather stations are cylinders which have a funnel-shaped opening whose area is 10 times that of the cross section area of the collecting tube. Thus, the depth of the water in the collecting tube is ten times the actual amount of rainfall. If the depth of the water in the collecting tube was 6″ for a certain period, what was the actual rainfall for that period?



2. During a period of seven years, the amount of snow which fell annually on Mount Rainier Park in Washington was as follows: 555 inches, 582 inches, 573 inches, 578 inches, 567 inches, 603 inches, and 551 inches. What was the average snowfall over the seven years?

3. During the period 1955–56, Mount Rainier Park had a record of 1000 inches of snowfall. It takes approximately 11 inches of snow to equal one inch of rain. Therefore, 1000 inches of snow equals how many inches of rain? Round your answer to the nearest tenth.

4. Louisiana has a higher average rainfall than any other state in the nation. The average rainfall in Louisiana is 55 inches per year. How many feet (to the nearest tenth) does 55 inches represent?

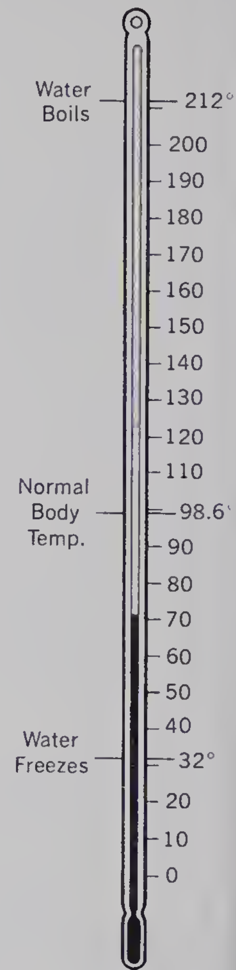
5. The average rainfall in New York City is 42 inches per year. Chicago has 9 inches less than New York. The rainfall in Chicago each year is what fraction of a yard?

6. Omaha has an average rainfall of 26 inches per year. Snow accounts for 3.5 inches each year. How much snow does it take to equal 3.5 inches of rain?

7. The average rainfall over the entire United States is 29 inches per year. Is it possible that the rainfall over the United States will never be exactly 29 inches for any given year?

8. Over a period of ten years, the annual rainfall in inches in Death Valley, California was: 1.96, 2.11, 2.33, 1.78, 2.01, 1.96, 1.82, 2.15, 1.96, 2.01. Was there more rainfall during the first 5 years or during the second? Find the average rainfall per year.

The daily temperature affects many of the things we do. We might go swimming or ice skating, or we might open the windows wide or close them and turn up the heat, all depending upon the outside temperature.



1. In the United States we measure temperature with a Fahrenheit thermometer. (See the Figure on the right and also on page 309.)
   a. What is the boiling point of water?
   b. What is the freezing point of water?

2. The hottest temperature ever recorded officially was in Libya, North Africa on September 13, 1922 when the thermometer read 136° *F*. How many degrees below the boiling point was this?

3. The temperature at Death Valley, California has reached 134° *F* which is the highest temperature recorded in the United States. The lowest temperature recorded was 69.7° *F* below zero at Rogers Pass, Montana. How many degrees difference is there between the two temperatures?

4. In the United States, temperatures lower than 60° *F* below zero have been recorded in only four other states: Colorado, Wyoming, North Dakota, and Idaho. The lowest temperature on record is 102.1° *F* below zero at the South Pole. How much colder than the coldest record temperature in the United States is this?

5. Some places in the United States have a fairly constant temperature. The average monthly temperature at San Francisco varies from 50° *F* in the coldest month to 62° *F* in the warmest month. This is a range of 12° from the higher to the lower. The average monthly temperature at St. Louis goes from a low of 33° *F* to a high of 81° *F*. What is the range in St. Louis?

6. The key to all weather forecasting is *air pressure* as measured by a *barometer*. A cubic foot of air weighs 0.0765 of a pound. What would be the weight of the air in a room that is empty of objects and is 10 feet wide, 15 feet long, and 9 feet high?

7. The blanket of air around the earth is about 700 miles thick. Above an altitude of 5 miles, however, the air becomes quite thin. The weight of the air causes pressure on objects on the earth. Why does this pressure decrease as you rise higher above sea level?

Can you believe that even at the equator people live only a few miles from places as cold as the North Pole? Changes in altitude affect the air pressure which in turn affects the temperature. To find a different temperature we need only to go up or down from the earth's surface.

In problems 1 to 8 consider the temperature at the surface of the earth as 70° F.

1. Temperature normally decreases about 3 degrees Fahrenheit for each 1000 feet we ascend into the atmosphere. What do you expect the temperature to be at 12,000 feet?

2. A jet plane is traveling at an altitude of 46,000 feet. What is the temperature outside the cockpit?

3. Recently the X–15 rocket plane made an altitude record at 126,000 feet. What was the temperature at this height, assuming that the temperature dropped 3° per thousand feet?

4. How far above Chicago would the temperature be 0° F?

5. The temperature at the North Pole sometimes reaches 100° F below zero. At what altitude above Chicago might the same temperature be reached?

6. In a mine shaft temperature increases 1° for each 60 feet we go below the surface of the earth. How much warmer is it a mile underground than it is on the surface?

7. How far beneath the surface would the temperature in a mine shaft reach the boiling point of water (212° F) in the unlikely event that one were built?

8. The "snow line" is the level below which the snow melts on the mountainside. If the temperature at Los Angeles is 70° F, how high up the mountainside would you go to reach 32° F, where the snow begins to melt?

9. At sea level the air pressure is 14.7 pounds per square inch. At an altitude of 95,000 feet the air pressure is only 0.3 of this. What is the air pressure at 95,000 feet?

10. At 62 miles altitude the air pressure is as low as the best vacuum that man can produce. This is how many miles above the altitude reached by the X–15 (Exercise 3)?

11. Residents of many cities in the west can drive short distances to find cooler weather. The elevation at Greeley, Colorado, is 4000 feet. At Estes Park, 50 miles away, the elevation is 10,000 feet. When the temperature at Greeley is 95°, what is the temperature at Estes Park?

Differences in air pressure cause movements of air which we call *wind.* They vary from gentle breezes to fierce hurricanes.

1. The *chinook* wind is a movement of air that is warmed as it moves down the eastern slope of the Rocky Mountains. In Canada it is recorded that a chinook wind raised the temperature from 5° F below zero to 54° F above zero over a 6-hour period. What was the average per hour increase in temperature during this interval?

2. In March, 1900, an unusual chinook raised the thermometer 33° F in 3 minutes in Montana. If the temperature was 5° F below zero when the chinook started, what was the temperature after 3 minutes?

3. The *monsoon* winds of southeastern Asia blow toward the land in summer bringing abundant rainfall. One spot in India receives 400 inches of rainfall a year. What is the average amount per day? Round your answer to the nearest tenth of an inch.

4. The record for rainfall in a 24-hour period occurred in the Philippine Islands when 3.83 feet of rain fell. How many inches (to the nearest inch) is 3.83 feet?

5. What was the average rainfall per hour during the 24-hour period (Exercise 4)?

6. The record rainfall in the United States was 38.7 inches in 24 hours. This happened in Yankeetown, Florida, in September, 1950. The average annual rainfall in Death Valley, California is 2 inches per year. How many years would it take Death Valley to receive as much rain as did Yankeetown during its record 24-hour period?

7. The strongest wind recorded at ground level in the United States is 231 miles per hour at Mt. Washington, New Hampshire. What was the speed in miles per minute? Round your answer to the nearest tenth of a mile.

8. The most feared wind is the *tornado.* In some tornadoes winds reach a velocity of over 300 miles per hour. In 1957, 1600 tornadoes were reported, but only 80 of them were reported to have caused property damage. What fraction of those reported caused damage?

9. *Fog* is another weather problem. The foggiest place in the United States is the coast of Maine which has 1554 hours of fog each year; 1554 hours is equal to how many 24-hour days?

10. Seattle, Washington has clear skies on an average of 80 days per year. Yuma, Arizona has 280 clear days per year. Yuma has how many times as many clear days as Seattle?

11. What per cent of the days are clear during the year at Yuma?

Oral Exercises

In working these problems, do not use pencil and paper unless it is necessary.

1. During three consecutive months Healdsburg received 3.5 inches, 4.5 inches, and 2.3 inches of rainfall respectively. What was the total rainfall for the three-month period?
2. Boise, Idaho had $2\frac{1}{2}$ times as much rainfall as Reno, Nevada during one month. If Reno's rainfall for the month was 2 inches, how much rain fell in Boise?
3. From noon to 7:00 P.M. the temperature dropped $14.3° F$ at Healdsburg. If it was $87.4° F$ at noon, what was the temperature at 7:00 P.M.?
4. Williamstown received its first snowfall of the year with $2\frac{1}{2}''$ on Monday, $3\frac{1}{2}''$ on Tuesday, and $4\frac{1}{4}''$ on Wednesday. If none of the snow melted, how much snow was on the ground after Wednesday?
5. The average temperature of Hanford was $2.3° F$ higher in June than in May. If the average temperature was $74.6° F$ in June, what was the average temperature in May?
6. The wind was blowing with a speed of 28 m.p.h. in Middleton at 2:00 P.M. By 10:00 P.M. the speed had dropped to one-fourth of the speed at 2:00 P.M. What was the speed at 10:00 P.M.?
7. Wellington had $3\frac{1}{4}''$ of rainfall in January. The rainfall at Madera during the same month was $2\frac{1}{8}''$. How much more rainfall did Wellington get than Madera?
8. Roseburg had $7''$ of snow on the ground on Sunday. On Monday $4\frac{1}{2}''$ more snow fell. During Tuesday, Wednesday, and Thursday $2''$ of the snow melted. On Friday and Saturday $3\frac{1}{2}''$ more snow fell. How much snow was then on the ground at Roseburg?
9. The total rainfall for the year at Cooperville was $38.4''$. What was the average monthly rainfall?
10. During a rainstorm at Centralia $\frac{1}{8}''$ of rain fell each hour for a 24-hour period. How much rain fell on Centralia during this period?
11. The normal annual rainfall at Brownsville is $21.6''$. If Brownsville received $\frac{2}{3}$ of the normal rainfall during one year, how much rain did Brownsville receive?
12. One day in February the highest temperature in Havre, Montana, was reported as 27°; the lowest was $^{-}13°$. What was the range in temperature during the day?

The steps for solving applied problems should be of help in working each of the following problems.

STEPS FOR SOLVING APPLIED PROBLEMS

1. Understand the problem.
2. Note what the problem asks for.
3. Look for hidden questions.
4. Estimate a reasonable answer.
5. Set up and solve the conditional statement(s).
6. Check your answer.

1. Miss Adams' mathematics class is planning an outdoor lunch. The class intends to serve $\frac{3}{4}$ of a cup of punch to each of 56 pupils. How many quarts of punch will the class need?

2. Mr. Erickson is going to put a fence around a rectangular pasture that is 20 rods long and 16 rods wide. How many feet of fence will he need?

3. During a recent track meet the longest javelin throw was 72.8 meters. How far is this? Give your answer in feet and inches.

4. There are 32 pieces of sheet copper piled on a shelf. The pile is 8″ high. How thick is each sheet of copper?

5. The first train from New York to San Francisco completed the trip in 6 days, $3\frac{1}{2}$ hours. Today the train trip takes 3 days, $12\frac{1}{2}$ hours. How many hours less does it take today?

6. Jerry needs 6 pieces of rope each 5′ 8″ long. How many feet of rope does he need altogether?

7. Jim is going to cut 6 pieces of steel from a rod 4′ long. Each piece will be 6.3″ long, and 0.2″ will be allowed for each cut. How long will be the piece of rod that is left?

8. On his four weekly tests last month, Harry made scores of 95, 85, 87, and 92. What was his average for the four tests?

9. A car traveled 184 miles on 11.3 gallons of gasoline. To the nearest mile, how many miles per gallon did it travel?

10. At 34.9¢ per gallon, find the cost of the gasoline used in Exercise 9.

11. How many inches are in 880 yards?

12. In a track meet, a certain athlete ran the mile in exactly 4 minutes. What was his average speed in miles per hour?

FINDING THE HIDDEN QUESTION

The exercises that follow will give you practice in finding and answering hidden questions in problems. For some of them you will need to make a sketch to be sure of using the correct data. Some of the problems have unnecessary data, and some do not. Watch your step.

1. A gallon of water weighs about $8\frac{1}{3}$ pounds. There are 4 quarts to a gallon and 2 pints to a quart. How many pounds will 32 pints of water weigh?

2. When Joe started on an automobile trip he filled the gasoline tank and set the odometer at zero. When he reached Middleton it registered 149 miles. It took 8.5 gallons to refill the tank. How many miles per gallon of gasoline did Joe get?

3. A machinist needed to select a steel rod that he can cut into 14 pieces each $3\frac{1}{4}''$ long for bolts. He must allow $\frac{1}{4}''$ for each saw cut. How long is the steel rod that he will need to get the 14 pieces?

4. If you cut 6 pieces each 4.5 inches long from a brass rod 34.5 inches long, allowing 0.125 inches waste due to each saw cut, how long is the piece that is left?

5. A brass rod 10 inches long measures 0.45 inches in diameter. It is cut into pieces each 2.125 inches long. If 0.15 inches is to be allowed for waste from each saw cut, how many pieces can be cut from the rod?

6. How long is the piece that will be left over (Exercise 5)?

7. A bar 2 m. long is divided into 20 equal pieces. Allowing 2 mm. waste for each saw cut, what is the total waste from the saw cuts?

8. How long will each piece be (Exercise 7)?

9. A machinist is planning to cut 21 pieces each 15 cm. long from a bar of aluminum 5 m. long. Allowing 3 mm. for each saw cut, how long is the bar that will be left after the order is filled?

10. A 40-oz. can of pineapple juice can be purchased for 40¢. A 16-oz. can costs 18¢. What is the cost per ounce of pineapple juice when purchased in the 40-oz. can?

11. In a certain factory employees are paid \$3.80 per hour up to 40 hours per week. In addition, they receive a bonus of 50% of their hourly pay for each hour overtime (over 40 hours). How much pay would an employee receive for working 48 hours in one week?

12. There are approximately 231 cubic inches in a gallon. About how many cubic inches are there in a pint?

13. In a machine shop, 84 lb. of $\frac{5}{8}''$ bolts were received in a shipment. How many bolts were received if each bolt weighs 12 oz.?

1. A magic square: You can make a magic square if you can answer correctly the questions below, and if you insert the answer to each question in place of its letter in the square.

A	B	C
D	E	F
G	H	I

Lay out the square with nine cells on a sheet of paper. When you have filled in the correct answers to these questions, the three columns, the three rows, and the two diagonals should each add up to 30.

a. How many cups are there in a quart?
b. How many inches are there in half a yard?
c. How many quarts are there in 2 gallons?
d. How many feet are there in $4\frac{2}{3}$ yards?
e. What per cent of a mile is 32 rods?
f. How many inches is 50% of a foot?
g. How many inches is 100% of a foot?
h. How many pints are there in 1 quart?
i. How many inches are there in $\frac{4}{9}$ of a yard?

2. Here is a harder one. Lay out a square with 9 cells lettered as above. Find each of the following ratios, and enter it in the cell with the same letter. It will be easier if you use decimals to represent each ratio. When the cells are filled, you should have a magic square. If you do not, find your mistake.

The ratio of:

a. 1 bushel to 2 pecks
b. 1 yard to 2 feet
c. 1 gallon to 1 quart
d. 1 yard to 8 inches
e. 1 mile to 128 rods
f. 1 pint to 1 quart
g. 36 inches to 1 yard
h. 42 inches to 1 foot
i. 1 yard to 1 foot

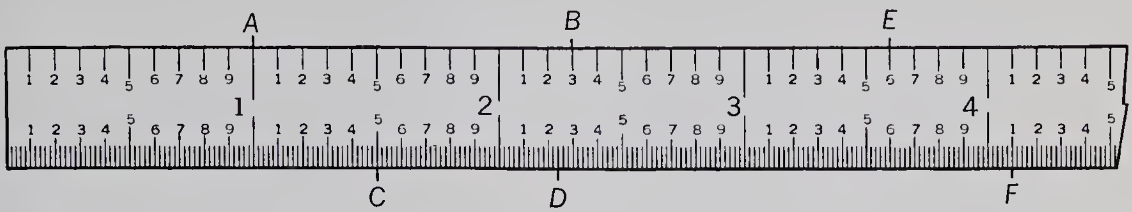
3. Lay out a square with 9 cells lettered as above. Enter the answer to each of the following in the proper cell.

a. Write $\frac{176}{528}$ in simplest form.
b. Jim wants to buy a set of golf clubs that will cost \$144. He now has \$6. This is what fraction of the purchase price?
c. At a year-end sale a book that had been priced at \$3.20 was sold for \$2.40. The reduction was what fraction of the original price?

Write the ratio, as a fraction, of:

d. 1 quart to 1 peck
e. $2\frac{1}{2}$ inches to 1 foot
f. 1 foot 9 inches to 2 yards
g. 4 hours to 1 day
h. 120 rods to 1 mile
i. 1 quart to 3 gallons

1. In the machine shop fractions of an inch are commonly expressed as decimals. Below is an illustration of a ruler used in the machine shop.

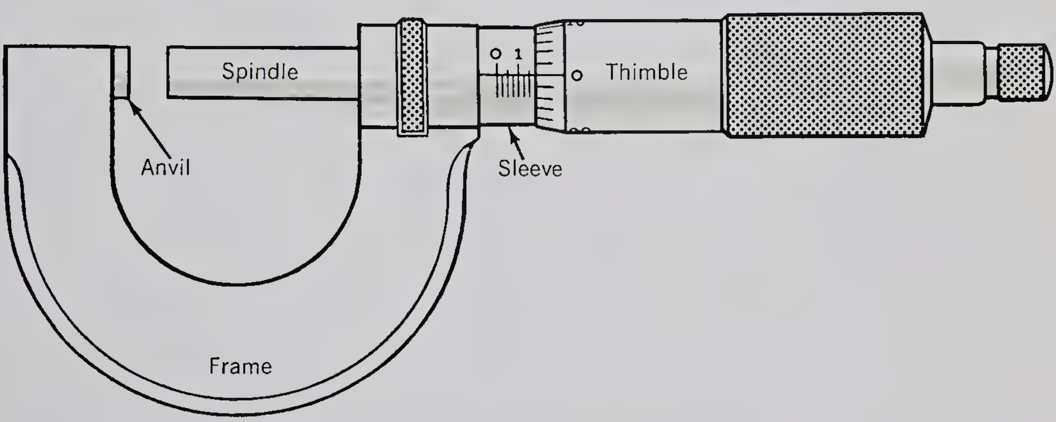


What is the distance from A to B? from C to D?

2. Between what two letters in the above Figure is the distance 1.86″?
3. Frequently it is necessary, in making measurements in the machine shop, to change from fractions, such as halves, fourths, and eighths, to decimals. Ordinarily, on the wall of the shop, there is a chart with a table of fractional equivalents. What is the decimal equivalent of each of these fractions?

a. $\frac{1}{8}''$	**c.** $\frac{3}{4}''$	**e.** $\frac{25}{32}''$
b. $\frac{5}{16}''$	**d.** $\frac{7}{8}''$	**f.** $\frac{9}{16}''$

4. The thickness of sheet metal is expressed in *gauge number*. Gauge number 1 is $\frac{9}{32}''$ thick. In the machine shop, to determine the gauge number you would measure the thickness of the sheet metal with a *micrometer caliper* (the Illustration below) that is marked off in decimal parts of an inch. Express the thickness of gauge number 1 as a decimal to the nearest hundredth.

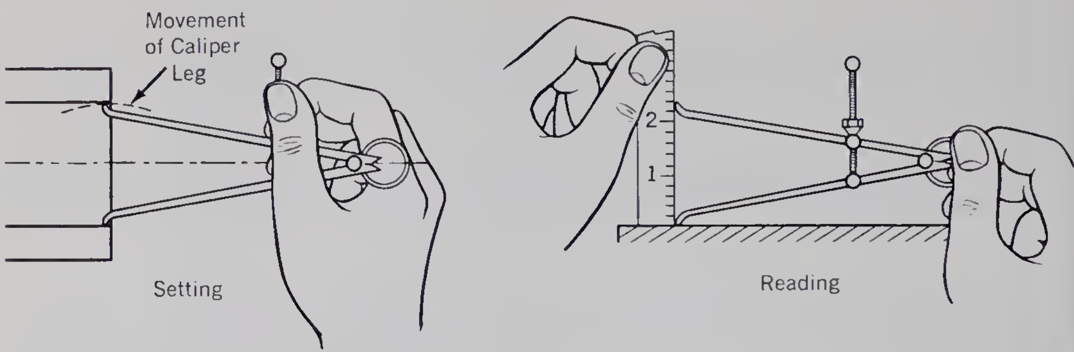


5. The thickness of other commonly used gauge numbers of sheet metal are:

No. 2, $\frac{17}{64}''$; No. 3, $\frac{1}{4}''$; No. 4, $\frac{15}{64}''$; No. 5, $\frac{7}{32}''$

Express each of these thicknesses as a decimal to the nearest hundredth.

6. The width of openings up to a few inches is measured by *inside calipers*. (See the Illustration below.) The caliper is first adjusted to the opening and then placed on a rule to read the measurement. What is the measurement in the Illustration?



7. To measure very small openings (thousandths of an inch) such as those between parts of a machine, a *thickness gauge* (sometimes called a *feeler gauge*) is used. (See Figure 1 below.) The gauge shown here has six *leaves*, although the number of leaves depends on what is to be measured. The machinist will use different combinations of leaves until he finds the ones that fit the opening he is measuring, then he will add up their values. Suppose he uses leaves a, b, and d. How large is the opening?

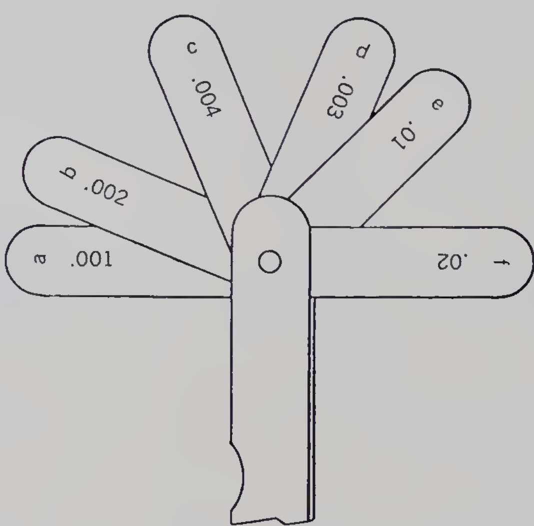


Figure 1

8. What is the size of the opening being measured if leaves c, d, and f are used?

9. Suppose the machinist wanted to adjust the opening to 0.035″. Which leaves would he use?

10. Which leaves would be used to adjust an opening that should measure exactly 0.029 inches?

11. A thickness gauge with nine leaves is shown in Figure 2 below. Which four leaves would you use to adjust to an opening of 0.114″?

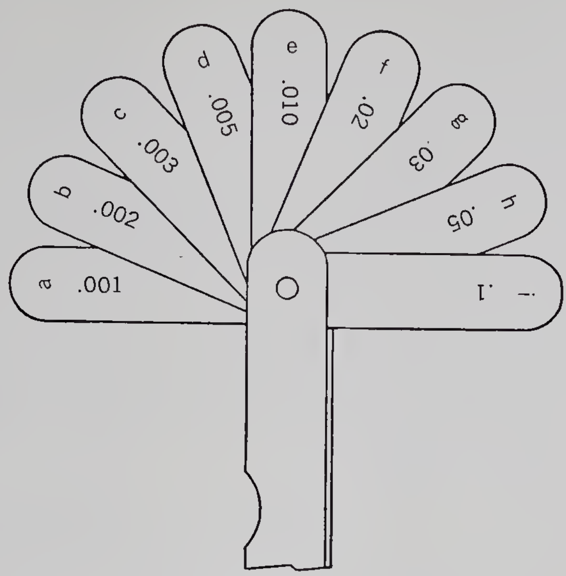


Figure 2

12. Which leaves on the gauge (Figure 2) would add up to 0.18″?
13. Suppose the machinist wished to adjust an opening to 0.019″. Which leaves of the gauge (Figure 2) would he use?
14. What thickness would leaves a, d, h, and i add up to?
15. Figure 3 is a drawing of a *spark plug*. What is the total number of inches in the length of the spark plug?

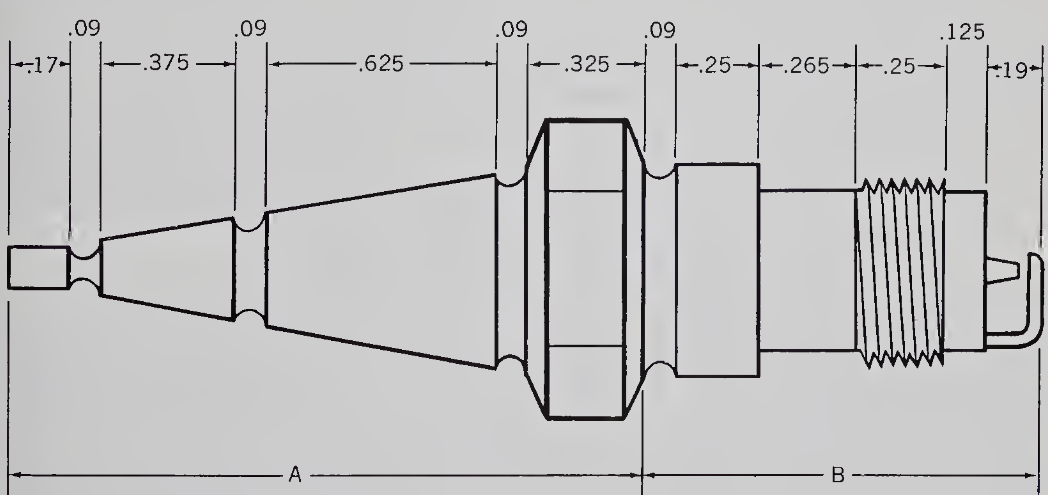


Figure 3

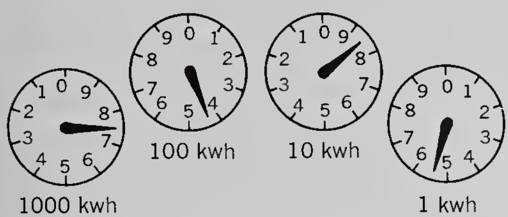
16. In Figure 3 what is the distance indicated by *A*?
17. What is the distance indicated by *B*?
18. The distance indicated by *A* is how much greater than the distance indicated by *B*?

It is remarkable how little the English-speaking people know about their own systems of measurement. Do you share this lack of information? Answer as many of these questions as you can without referring to the tables in the chapter or in the back of the book. Then look up the answers to the rest.

1. How many grains are in an apothecaries' ounce?
2. How many yards are in one rod? How many rods are in a mile?
3. If a pint of a certain material weighs 1.5 pounds, what will a cubic foot of the material weigh?
4. A "liquid ounce" is a measure of capacity. How much does a liquid ounce of water weigh?
5. Which is larger, and by how much: a dry quart or a liquid quart?
6. How many quarts are in a barrel?
7. How much does a barrel of water weigh?
8. How many square feet are in an acre?
9. A city block was laid out to measure 100 yards square. To the nearest acre, how many acres does it contain?
10. Each lot is to be 60 feet wide and 150 feet deep. How many lots are in an acre? Round your answer to the nearest whole number.
11. The capacity of a reservoir in an irrigated region is commonly reported in *acre-feet,* which is the amount of water that will cover an acre to a depth of one foot. If a reservoir has a capacity of 1 million acre-feet, how many barrels will it hold?
12. Which is greater, and how much: a long ton, or a metric ton?
13. An Imperial gallon, used in England and Canada, is the volume of 10 pounds of water at a specified temperature and pressure. How many Imperial gallons are in one cubic foot?
14. How many U.S. gallons are in one Imperial gallon?
15. A United States resident is driving his car in Canada on vacation. The gasoline tank has a capacity of 20 U.S. gallons and registers $\frac{1}{4}$ full. He asks the filling station attendant to put in 15 gallons. After how many Imperial gallons are put in, will the tank overflow?
16. If a peck of wheat weighs 15 pounds, what will a cubic foot of wheat weigh?
17. The grain is the same in avoirdupois and troy weight. The pound avoirdupois is 7000 grains and is 16 ounces. The pound troy is 5760 grains and is 12 ounces. How many grains are there in each ounce? Which ounce is heavier?

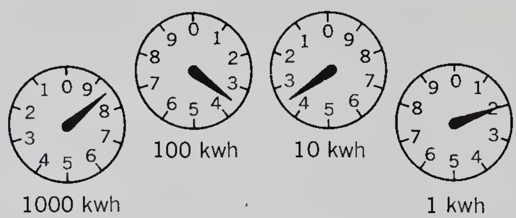
Electricity for lights and other electrical appliances is measured by an electric meter. The unit of measure is the *watt-hour* or the *kilowatt-hour* (kwh), which is 1000 watt-hours. For example, if a 1000-watt heater is used for one hour, or a 100-watt bulb is burned for 10 hours, 1 kilowatt-hour of electricity is used. (1 kilowatt equals 1000 watts.) On most of the electrical appliances used in the home you can find a label that tells the amount of electricity each uses. Thus, an electric iron labeled "500 watts" uses a kilowatt-hour every two hours.

1. In reading an electric meter, always read on each dial the numeral that the hand has just passed. Study the dials below and explain why you should read each dial this way.



The dial at the left reads 7485.

Read the meter at the right.



2. On October 1, the meter in Mr. Anderson's house read 4642 (kwh). On November 1, the reading was 4677 (kwh). How many kwh were used?

3. Draw two sets of dials. On one set show the positions of the hands on October 1, and on the other show the position of the hands on November 1.

4. The following rates are charged in Mr. Anderson's home town:

First 20 kwh	9¢ a kwh
Remainder	5¢ a kwh

Find the cost of the electricity used by the Andersons during October.

5. An electric fan is marked "40 watts." How many hours will it run on one kwh of electricity?

6. At 5¢ a kwh, how much per hour does it cost to operate the fan?

7. A 100-watt light bulb is left burning for eight hours. At 5¢ a kwh, how much did the electricity cost?

8. What would it cost to operate the fan in Exercise 5 for one hour if electricity costs 10¢ per kwh?

9. A large part of the convenience and comfort of our homes today is due to the large number of electrical appliances of various kinds. How many of the appliances listed below have you used?

a. Refrigerator (200 watts)
b. Mixer (80 watts)
c. Electric fan (100 watts)
d. Television (800 watts)
e. Radio (75 watts)
f. Automatic fryer (1100 watts)
g. Heater (1320 watts)
h. Percolator (400 watts)
i. Toaster (900 watts)
j. Light bulb (75 watts)
k. Electric clock (1 watt)
l. Drill (250 watts)

10. The number of watts ordinarily used by each kind of appliance is shown with the above list. Which three use the most? Which three use the least?

11. Henry Smith wished to calculate the cost of the electricity used in doing the laundry at home during one week. The washing machine is marked "360 watts." The electric iron is marked "1000 watts." He found that the washing machine was used for 2 hours and the electric iron was used for 4 hours. If electricity costs 8¢ per kwh, what was the cost of the electricity used?

12. A 100-watt light bulb in the garage was negligently left burning for ten hours. At 5¢ per kwh, what was the cost of the electricity used?

13. An electric refrigerator uses electricity only while the cooling mechanism is in operation. Mabel checked and found that in their 200-watt refrigerator the mechanism was in operation one-fifth of the time, during a 24-hour day. Using this as an average, and a rate of 4¢ per kwh, what was the daily cost of the electricity?

SPECIAL PROJECTS

1. Bring in a list of five electric appliances commonly used around the home. State the amount of electricity each uses in watts or kilowatts if you can find it on the appliance.

2. Appliances that use electricity for producing motion, such as a vacuum cleaner, usually use less electricity than those that use it for heating purposes, such as a toaster. Show for which of the purposes each of the appliances on your list is used. Is the statement above generally true?

3. Determine what it costs per hour to use each appliance on your list. Use the electricity rates of your community.

4. If a mistake is made in reading an electric meter one month, and the consumer is overcharged, he will be undercharged by the same amount the next month even though the company does not discover the mistake. Explain how this occurs.

Water is one of our most common materials because we use it in many ways. Most uses of water require measurement. As an electric meter is used to measure the amount of electricity used, a water meter is used to measure the amount (in cubic feet) of water used.

1. In chemistry experiments and in some cooking recipes water is measured by the drop. It takes 25 drops of water to fill a teaspoon. How many drops are there in a tablespoon?

2. How many drops of water are there in a cup?

3. Tommy noticed that 20 drops per minute escaped from the leaky faucet in the kitchen. How many cups of water were wasted each hour?

Liquid Measure

3 teaspoons	=	1 tablespoon
16 tablespoons	=	1 cup
2 cups	=	1 pint
1 pint	=	16 ounces (liq.)
2 pints	=	1 quart
4 quarts	=	1 gallon
7.5 gallons	≈	1 cubic foot

4. Industry has many uses for water. Tire manufacturers use 600 gallons of water to produce one tire. How much water is required to manufacture a set of 5 tires?

5. A gasoline refinery uses 15 gallons of water to produce 1 gallon of gasoline. An average family uses 632 gallons of gasoline per year. How much water does the refinery use to produce 632 gallons of gasoline?

6. When large amounts of water are used in irrigation projects, the water is measured in acre-feet. Remember! An acre-foot is the amount of water needed to cover an acre of surface to a depth of one foot. How many cubic feet are in one acre-foot?

7. Local water in one western community costs about $20 per acre-foot. It costs $65 per acre-foot to bring water from the Colorado River to Los Angeles. The cost of the local water is what fraction of the cost of the river water?

8. How many gallons, to the nearest gallon, are there in 27 cubic feet?

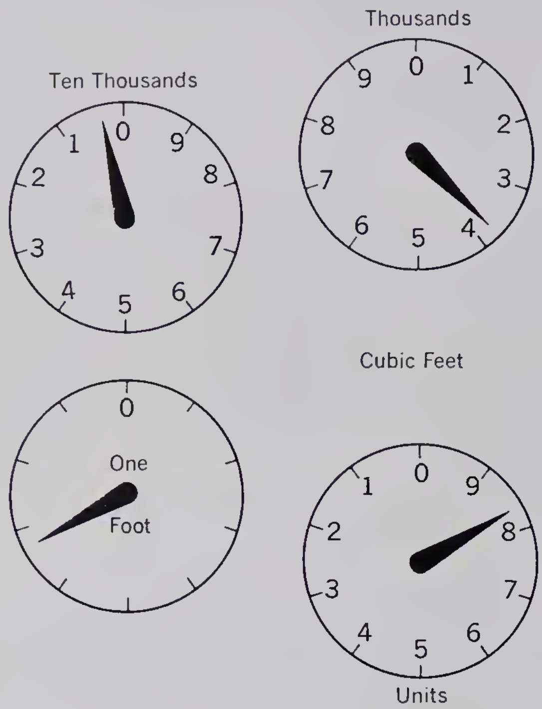
9. About how many gallons are there in one acre-foot?

SPECIAL REPORT

Driving carefully and at moderate speeds helps save our nation's supply of water. Can you explain how? What are some other ways you can suggest for conserving water?

The water used in the home of Jack Adams is furnished from a city supply tower. The water used is measured by a water meter installed in the home. Consequently, a water bill is sent to Mr. Adams every month. Jack Adams tried to check the bill.

He found that the water meter had six dials as shown in the following Illustration.



Jack said the meter read 3558 cubic feet. Jack disregarded the dial marked "one foot" as that is only to detect leaks. First, he read the dial marked "units." In the same way he read the dials marked "tens," "hundreds," "thousands," and "ten thousands" in consecutive order.

1. Did Jack read the meter correctly?

2. If the previous reading a month ago was 1276 cubic feet, how many cubic feet had they used during that period?

3. The rate was 50¢ per thousand cubic feet. Find the amount of the bill.

4. At \$1.50 per 1000 cubic feet, what would be the cost of 3800 cubic feet of water?

5. During one summer month the Adams family had a water bill of \$6.25. At 50¢ per 1000 cubic feet, how many cubic feet of water did the family use during this month?

6. Jack checked the water bills for the entire previous year and found they totaled \$42.60. What was the average monthly bill?

7. How many cubic feet of water was used by the Adams family during the year?

8. Helen kept track of the meter readings at her home for a six-month period. They were as follows:

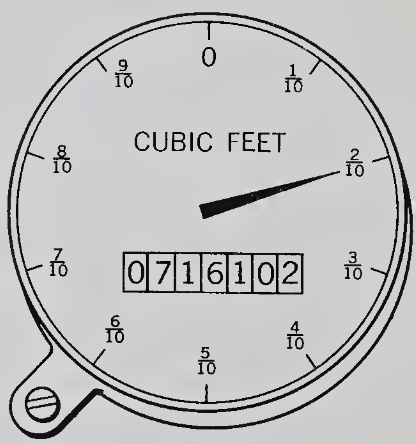
Date	*Reading*
September 1	8,000 cu. ft.
October 1	11,600 cu. ft.
November 1	14,700 cu. ft.
December 1	18,200 cu. ft.
January 1	22,100 cu. ft.
February 1	25,300 cu. ft.
March 1	28,500 cu. ft.

How many cubic feet of water were used each month during this period? If the rate was $1.50 a thousand cubic feet, what was the cost of water each month?

9. In some cities water meters are not used and each family is charged a "flat" rate. Which would have been cheaper for Helen's family, a flat rate of $3.00 a month or a rate of $1.00 per 1000 cubic feet?

SPECIAL PROJECTS

1. The water meter pictured on the right is a type having only one dial instead of several. Explain how you would read it. What advantages can you see from this type of meter? Do you see any disadvantages? What type is used in your city?



2. Find out about the water supply in your city:
   a. Where does the water come from?
   b. What rates are charged?
   c. If meters are used, how often are they read?

3. Gas is sold by the cubic foot. It is measured by a meter that can be read like an electric meter. You can readily learn to read your gas meter. If you can find out the cost per cubic foot of gas, you can determine the per hour cost of using your gas appliances. Read your gas meter at home and bring the reading to class.

4. Read your meter a week from today and find how many cubic feet of gas you used during the week. Determine the cost of the gas used.

Part One

1. **a.** If a mile is 5280 feet, how many yards are in a mile?
   b. If an inch is 2.54 centimeters, how many centimeters are there in one foot? (Round your answer to the nearest tenth.)
   c. How many inches are there in 50 centimeters? (Round your answer to the nearest tenth.)
   d. If we say that a kilometer is about $\frac{5}{8}$ of a mile, approximately how many kilometers are there in 200 miles?
   e. How many inches are contained in a mile?
   f. If the distance from A to B is 28,548 feet, how many miles is this? (Round your answer to the nearest tenth.)
2. **a.** If a ton is 2000 lbs., what fraction of a ton is 800 lbs.?
   b. How many pounds is 72 ounces?
   c. What is the ratio of one ounce to one pound? one pound to one ton?
   d. If a pound is about 0.453 kilograms, approximately how many kilograms are there in one ton?
   e. To the nearest tenth, a kilogram is how many pounds?
   f. Fred weighs 48 kilograms. To the nearest tenth, how many pounds does he weigh?
3. **a.** If a gallon equals 4 quarts, and 1 quart equals 2 pints, how many pints are there in 6 gallons?
   b. What is the ratio of a pint to a gallon?
   c. There are approximately 7.5 gallons to a cubic foot. How many gallons will a tank 3 feet long, 2 feet wide, and 1 foot deep contain?
   d. If a quart is approximately 0.95 liters, how many liters are there in one cubic foot?
   e. A meter is approximately 39.37 inches. How many meters are there in 100 yards? (Round your answer to the nearest meter.)
   f. If a cubic foot of water weighs about 62.5 lbs., what does one gallon of water weigh? (Round to the nearest tenth of a pound.)
4. **a.** A meter is approximately 39.37 inches long. To the nearest tenth, this is how many inches?
   b. A kilometer is how many yards? how many feet?
   c. To the nearest hundredth, a kilometer is what part of a mile?
   d. How many meters, to the nearest meter, are in a mile?
   e. How many kilometers, to the nearest hundredth of a kilometer, are in a mile?

Part Two

1. Write the letters a through f on a sheet of paper. Then after each letter write the error of measurement for the corresponding statement that follows:
   - **a.** The earth revolves about the sun in 365 days.
   - **b.** The car averages 18 miles per gallon of gasoline.
   - **c.** He can run 100 yards in 9.6 seconds.
   - **d.** You can cross the continent by air in $4\frac{1}{2}$ hours.
   - **e.** The speedometer reads 38 miles per hour.
   - **f.** I can walk to school in 16 minutes.

2. Write the letters a through g on a sheet of paper. After each letter indicate by T or F whether each of the following statements is true or false.
   - **a.** A yard is longer than 100 cm.
   - **b.** A liter is approximately 1 gallon.
   - **c.** 1000 meters is approximately 1 mile.
   - **d.** To the nearest tenth of a mile, 1 kilometer is 0.6 miles.
   - **e.** A change of one degree Celsius is a greater change in temperature than one degree Fahrenheit.
   - **f.** A room temperature of 98 degrees Celsius is warm but bearable.
   - **g.** A meter is slightly greater in length than 3 feet.

3. Write the letters a through i on a sheet of paper. Then write the missing word or numeral after each letter that will make the corresponding statement true.
   - **a.** A meter is __?__ centimeters.
   - **b.** A pound is __?__ ounces.
   - **c.** An inch is $\frac{1}{12}$ of a __?__.
   - **d.** Seventy-two feet is the same distance as __?__ yards.
   - **e.** Twenty-eight quarts is __?__ gallons.
   - **f.** Six miles is __?__ feet.
   - **g.** One liter is about __?__ pints.
   - **h.** A kilogram is __?__ grams.
   - **i.** The boiling point on the Celsius scale is __?__ degrees.

4. Write the letters a through h on a sheet of paper. After each letter write the ratio of the first measure to the second in each of the following pairs.
   - **a.** ounce; pound
   - **b.** foot; rod
   - **c.** quart; bushel
   - **d.** yard; mile
   - **e.** rod; yard
   - **f.** ton; ounce
   - **g.** kilometer; meter
   - **h.** millimeter; meter

Part Three

1. The speedometer of Fred's European car read 100 kilometers per hour. If the speed limit was 55 miles per hour, and a kilometer is about 0.62 miles, by how much was Fred exceeding the speed limit? Express your answer in miles per hour.
2. The rainfall at Marysville was 34.8 inches during one year. What was the average rainfall per month?
3. A 4-ounce bottle of vinegar cost 19¢. At the same rate, what would a bottle of vinegar weighing $1\frac{1}{2}$ pounds cost?
4. A plot of ground consisting of $10\frac{1}{2}$ acres is to be subdivided into lots each measuring $1\frac{3}{4}$ acres. How many lots can be obtained?
5. On a trip of 485 miles, Mr. Wilson's car used 31 gallons of gasoline. What mileage did his car give him on the trip? Give your answer to the nearest mile.
6. Mary used 24 inches from a roll of ribbon 3 yards long. How many inches of ribbon remained?
7. How much cheaper is it to buy 40 pounds of potatoes at a price of 4 pounds for 49¢ than at a price of 1¢ per ounce?
8. If an inch is 2.54 centimeters, how tall is Bill in meters if he measures 5′ 10″? (Round your answer to the nearest hundredth.)
9. A certain triangle has three sides of equal measure. If each side measures 4.74 centimeters, what is the perimeter of the triangle?
10. From 7:00 A.M. to 11:00 A.M. the temperature rose 36.3 degrees Fahrenheit at Huntsburg. If the temperature at 7:00 A.M. was 11.3 degrees below zero, what was the temperature at 11:00 A.M.?
11. Bill cut a board 2 yards long into 5 pieces of equal length. What was the length of each piece? (Do not consider the loss due to cutting.)
12. A gallon of apple cider cost $1.20. At this rate, what was the cost per pint?
13. On Christmas Day the highest temperature recorded in Spokane, Washington, was 22°. The lowest temperature for the day was $^{-}18°$. What was the range in temperature for the day?
14. From Nogales, Mexico, to Mexico City is 1196 kilometers. This is how many miles, to the nearest mile?
15. The length of a room is reported as 35 feet $8\frac{3}{4}$ inches.
    - **a.** Explain the meaning of the measurement.
    - **b.** What is the error of measurement?
    - **c.** What is the relative error of measurement?
16. A liter is approximately 1.06 quarts. How many liters are in 10 gal.?

CHAPTER NINE

MORE ABOUT PROBLEM SOLVING

WORDS TO WATCH FOR

approximation	*proportion*
extremes	*ratio*
means	

Problem solving is a typical part of our everyday activities. Whether planning an automobile trip, making arrangements for a basketball game, deciding whether you can afford to buy a bicycle, or estimating the correct change you should receive after making a purchase, the ability to solve problems is a necessity.

The skill to solve simple problems without using a pencil and paper is important for several reasons.

In the first place, most of the problems you will need to solve do not call for a precise answer. You do not need to know, to the fraction of a gallon how much gasoline you will use on a trip, or to the exact minute how long it will take to get there, or to the nearest cent how much it will cost.

In the second place, you probably will not have paper and pencil at hand when you want to calculate what change you should receive after making a purchase. Fortunately, the skill needed to make a close approximation can be developed readily.

Finally, in solving problems that call for a precise answer, it is important to be able to estimate in advance what a "reasonable" answer should be in order to serve as a check after you have calculated the precise answer with pencil and paper. This, you may recall, is Step 4 in the steps for solving applied problems.

All these reasons point to the importance of learning to solve problems "mentally," obtaining the precise answer when the numbers are simple, and arriving at a useful approximation when they are not. A useful procedure, in the latter case, is to simplify the calculations by rounding one or more of the numbers to make them suitable for computation without pencil and paper.

EXAMPLE

A plane leaves San Diego for Seattle at 9:30 A.M. The distance is 1135 miles, and the pilot plans to travel at 400 miles per hour as an average speed for the entire distance including stops. At what time does he expect to arrive in Seattle?

First round the numbers so that you can deal with them without a pencil. If you round 1135 to 1200, the computation becomes easy. You know that your approximation will be *a little more than* the precise answer you would obtain through written computation with the numbers as given.

Thus you estimate that the trip will take about 3 hours (1200 ÷ 400), and arrival time should be 12:30 P.M. You know that 3 hours is an overestimate because you rounded 1135 "up." The exact time of expected arrival, calculated from the figures as given, is 12:20 P.M. Allowing for minor delays, the difference between the estimated and exact calculations is too small to be important.

In the exercises below, do the following:

(1) Write an estimate of a reasonable answer by using rounded numbers.
(2) State whether your estimate is more than or less than the precise answer.
(3) Calculate the exact answer.

1. Chickens are on sale at 39¢ per pound. What is the price of a $3\frac{1}{2}$-pound chicken?

2. Mr. White's car averages 16 miles to a gallon of gasoline. How many gallons will he use in traveling 300 miles?

3. How much will be saved by purchasing a 3-pound can of coffee for $1.95 rather than 3 one-pound cans at 68¢ per pound?

4. Mr. Casper earns $530 per month. He spends $\frac{1}{9}$ of this on transportation. Approximately how much per month is this?

5. The boys who play in the backfield on the Westlake High School football team weigh 159, 183, 162, and 177 pounds respectively. What is the total weight of the backfield?

6. A jet passenger plane flew at an average speed of 590 miles per hour for $3\frac{1}{4}$ hours. How far did it travel?
7. A farmer took a truckload of 31 pigs to market. The total weight of the pigs was 6105 pounds. What was the average weight per pig? Round your answer to the nearest pound.
8. Mr. Adams paid $58.90 for a set of four tires. What was the average cost per tire?
9. Bob plans to purchase a motorcycle priced at $269.95. He has saved $131.25. How much more does he need in order to buy it?
10. What will 6 basketballs cost at $8.90 each?
11. Merill Junior High School has 214 pupils in the seventh grade, 206 pupils in the eighth grade, and 217 pupils in the ninth grade. What is the total enrollment for the three grades?
12. Jim purchased 5 gallons of gasoline at 31.9¢ a gallon. How much change should he receive from a $5 bill?
13. A Boy Scout troop hiked 16 miles in $4\frac{1}{2}$ hours. What was the troop's average rate in miles per hour?
14. Jane plans to cut ribbons each $3\frac{1}{2}$ inches long from a piece $26\frac{3}{4}$ inches long. How many ribbons will she get?
15. The Andersons are planning to leave at 8:30 A.M. to drive to Elmtown which is 237 miles away. They plan to average 40 miles per hour. At what time do they intend to arrive in Elmtown?
16. Golf balls are on sale at 45¢ each. What is the price of 16 golf balls at the sale?
17. Jim and his father used 31 gallons of gasoline on a recent trip. The car averaged 15 miles to the gallon. How many miles did they travel?
18. The income of the Jensen family is $7300 per year. If $\frac{1}{4}$ of the income is spent for rent, how much per month is this?
19. The distance by air from New York to Los Angeles is 2475 miles. The flight takes $4\frac{1}{2}$ hours. What is the average speed, in miles per hour, of the plane?
20. A truckload of wheat weighs 5340 pounds. A bushel of wheat weighs 60 pounds. How many bushels of wheat are on the truck?
21. The price of a tire is regularly $48.20. At a sale the price of 4 tires is $144.60. How much per tire is the reduction?
22. Jim is saving to purchase a pair of baseball shoes for $24.50. He has saved $15.60. How much more does he need to save?
23. When Arthur paid for 9 gallons of gasoline at 32.9¢ per gallon he received $2.04 in change. If this was the correct change, what was the value of the bill he gave the attendant?

The first step in the problem-solving process is to be sure you understand the problem. Careless reading is a common source of difficulty in problem solving. These exercises are designed to give you practice in careful reading of problems.

Which of the sentences, a–d, after each statement, 1–10, gives information that is provided in the statement, or can be calculated from it?

1. Jim purchased 8 gallons of gasoline at 31.9¢ a gallon. The tax was 11¢ per gallon.

a. Jim purchased 31.9¢ worth of gasoline.
b. Gasoline sells for 11¢ per gallon. **c.** Jim paid 88¢ in taxes.
d. Over one-half of what Jim paid went for taxes.

2. In last Saturday's football game, the quarterback for Ames High School threw 18 passes. He completed ten of them. His passes gained 160 yards.

a. The quarterback completed $\frac{5}{9}$ of his passes.
b. His average was 1.6 yards per pass.
c. He completed more passes than the opposing quarterback.
d. He threw a total of 28 passes.

3. A swimming pool is 8 feet deep at the deepest spot and 2 feet 10 inches deep at the shallow end. The pool is 130 feet long and 22 feet wide.

a. The pool is square.
b. The pool is 8 feet 10 inches deep. **c.** The pool is 130 feet long.
d. The pool is too shallow for swimming.

4. A new 5-cycle automatic washing machine can be purchased for $10 down and payments of $3.30 per week.

a. The weekly payments on the washing machine are $5.
b. The weekly payments last for 33 weeks.
c. The number of weekly payments is not stated.
d. The weekly payments decrease after the down payment is made.

5. Mr. Johnson received $2500 for a short story that he submitted to a weekly magazine. He immediately spent one-half of the money for a trip to New York.

a. Mr. Johnson spent a week in New York.
b. Mr. Johnson spent $2500 for a trip to New York.
c. Mr. Johnson spent one-half of $2500 for a trip to New York.
d. Mr. Johnson spent $50 for a trip to New York.

6. There are 152 national forests in the United States. The forests are divided into districts. In charge of each district is a forest ranger. There are about 800 district rangers.

   a. The United States has 152 national forests.
   b. There are about 800 forests in the United States.
   c. There are 152 forest rangers in the United States.
   d. Each national forest covers 800 acres.

7. The Jones family traveled 3200 miles on their four-week vacation. They spent $60 for gasoline, $7.50 for oil, and $26.50 for car repairs, lubrication, and tire repair.

   a. The Jones family spent $7.50 for gasoline on their vacation.
   b. The Jones family spent $93 on their vacation.
   c. The Jones family spent more for repairs than they did for gasoline.
   d. The Jones family had a four-week vacation.

8. Nancy typed an average of 55 words a minute in the last typing test. In a shorthand test, she took an average of 95 words a minute. She expects to increase her typing to 70 words per minute and her shorthand to 120 words per minute before the end of the semester.

   a. Nancy can type an average of 120 words per minute.
   b. Nancy typed an average of 55 words per minute in the last typing test.
   c. Nancy expects to type an average of 95 words per minute before the end of the semester.
   d. Nancy expects to type an average of 120 words per minute before the end of the semester.

9. John entered three events in last week's track meet. He ran the 100-yard dash and finished fourth. He won the 220-yard dash and placed third in the broad jump.

   a. John won the 100-yard dash.
   b. John won three events in last week's track meet.
   c. John did not win the 100-yard dash.
   d. John won the broad jump.

10. The Lowe family had lunch at a drive-in restaurant on the way to the beach. The cost of the lunch was $3.15 plus 13¢ tax. Mr. Lowe tipped the waitress 50¢.

   a. The Lowes had hamburgers for lunch.
   b. Mr. Lowe left a tip of 13¢.
   c. The Lowes were on their way to the mountains.
   d. The cost of the Lowe's lunch including tax was $3.28.

How skillful are you in using the problem-solving steps? In working the problems which follow be sure to use each step on each problem. See how many of the problems you can solve without the use of pencil and paper.

STEPS FOR SOLVING APPLIED PROBLEMS

1. Understand the problem.
2. Note what the problem asks for.
3. Look for hidden questions.
4. Estimate a reasonable answer.
5. Set up and solve the conditional statement(s).
6. Check your answer.

EXAMPLE

Yesterday, 12 of the pupils in Miss Erickson's class had perfect papers. This is $\frac{1}{3}$ of her class. How many pupils are in her class? Let n represent the number of pupils in her class.

The conditional statement is:

$$\begin{aligned} 12 &= \tfrac{1}{3} \times n \qquad & p &= x \times y \\ 12 \div \tfrac{1}{3} &= n & p \div x &= y \\ 12 \times 3 &= n \\ 36 &= n \end{aligned}$$

(Can you do this without using a pencil?)

1. The width of Mr. Henderson's cornfield is equal to one-half of the length. The width of the field is 40 rods. What is the length of the cornfield?
2. One-fourth of Mrs. Smith's class was on the honor roll last week. If 7 pupils were on the honor roll, how many pupils are in the class? HINT: We know that $\frac{1}{4}$ of the class represents 7 pupils. We want to find the number that $\frac{4}{4}$ represents since $\frac{4}{4}$ represents the whole class.
3. The secretary of the Mathematics Club announced that $\frac{1}{10}$ of the members were absent last week on account of bad weather. If 6 members were absent, how many pupils belong to the club?
4. It is 20 miles from Johnstown to Ellendale. This distance is equal to $\frac{1}{3}$ of the distance from Johnstown to Baldwin. How far is it from Johnstown to Baldwin?

5. The enrollment of the Lincoln High School has doubled in the last six years. Six years ago the enrollment was 400. What is the present enrollment?

6. Mike says that $\frac{1}{4}$ of his problems were incorrect on the test last Tuesday. If he had 4 problems incorrect, how many problems were there on the test?

7. Lois sells subscriptions to a periodical. Last week she earned \$15. This is equal to $\frac{1}{5}$ of the total amount paid for subscriptions. How much did the subscriptions total?

8. In Miss Jensen's class, $\frac{1}{4}$ of the pupils have dogs. If 8 pupils have dogs, how many pupils are in the class?

9. George and Henry together purchased a boat on the lake where they fish. George paid $\frac{3}{4}$ of the cost of the boat. If his share was \$12, how much did the boat cost?
   HINT: If $\frac{3}{4}$ of the cost is \$12, how much is $\frac{1}{4}$ of the cost? How much is $\frac{4}{4}$ of the cost?

10. Mary says that she saves $\frac{3}{8}$ of what she earns each Saturday working at the supermarket. She saves \$1.20. How much does she earn?
    HINT: What is $\frac{1}{8}$ of her salary?

11. Mike and his father were driving from Centerville to Hillsdale. When they had gone 72 miles they had traveled 0.6 of the distance. How far is it from Centerville to Hillsdale?
    HINT: What is 0.1 of the distance?

12. At the Eagle Drug Store all prices are reduced $\frac{1}{5}$ for the After-Christmas sale. Jane purchased a fountain pen on sale for \$1.60. What is the regular price of the pen?

13. When the Nelsen family left home on a 430-mile trip to Ellsworth the mileage on the odometer read 25,173. When they stopped for lunch it read 25,345. How many miles farther was it to Ellsworth?

14. When Arthur, Jim, and Harry went on a fishing trip they agreed to divide their expenses equally after returning home. They kept a record of expenditures and found that Arthur had paid $\frac{1}{3}$ of the expenses, Jim $\frac{2}{5}$, and Harry the rest which was \$4. How much did each boy pay when they divided the expenses equally?

15. Joe is earning money to pay his college expenses. He has been able to save $\frac{5}{7}$ of what he earned. So far he has saved \$475. How much has he earned?

16. The number of girls in Wilson High School is 67 more than the number of boys. There are 562 girls. What is the enrollment?

17. The width of a city lot is $\frac{3}{5}$ of the length. The width is 60 feet. What is the length?

You should learn to obtain the information you need from a variety of sources — paragraphs, graphs, tables, maps, etc. Below is a table showing how the use of water by each person in this country has increased since the nation was formed. By bringing water into desert regions as well as industrial areas, the productivity of our nation has vastly increased. Finding new sources of water has been a continual problem as our population and industry have expanded.

TOTAL DAILY WATER NEEDS FOR EACH PERSON

Date	*Gallons Used*	*Date*	*Gallons Used*
1776	5	1900	135
1800	33	1950	200
1850	65	1975 *	400

* *Estimate*

1. What was the total number of gallons of water per person used daily in 1800?
2. How much water per person was used daily in 1900?
3. How much did the use of water per person increase from 1800 to 1900?
4. How many times as much water was used in 1900 as in 1800? Round your answer to the nearest tenth of a gallon.
5. The water used per person in 1850 is what fraction of the amount used in 1950?
6. How much increase per person is expected from 1950 to 1975?
7. How long did it take for the use of water to increase from the amount used in 1776 to 40 times that amount?
8. What was the increase in the use of water per person from 1900 to 1950?
9. In which fifty-year interval among those shown in the table was the rate of increase in the use of water greatest?

SPECIAL REPORTS

1. List 10 uses for water around your house.
2. Which of these uses are new within the past 25 years? How does this help explain the increased use of water?
3. Report on a business or industry that uses a large amount of water. Explain what the water is used for.
4. One new development that increases the per capita use of water is air conditioning. How is water used in air conditioning? How would this affect the use of water in a large city like New York?

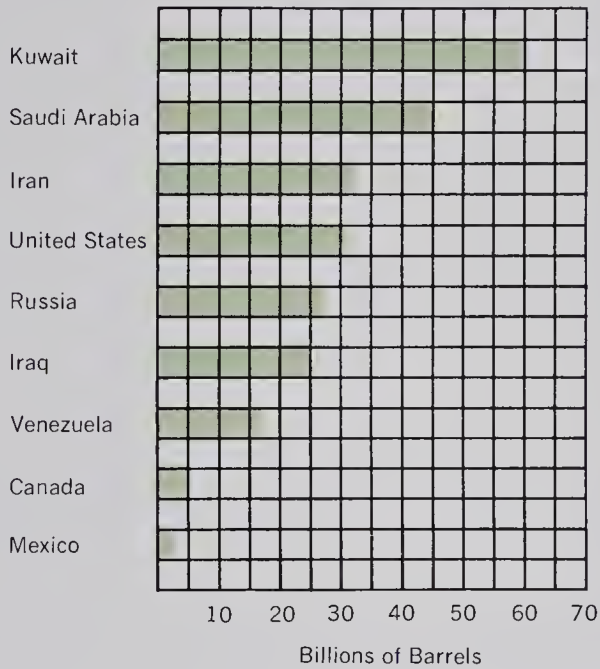
OBTAINING INFORMATION FROM A GRAPH

Sometimes the information you need to solve problems is presented in a graph. These exercises will give you practice in obtaining information from a graph.

The bar graph below shows the amount of petroleum left in the ground according to the best estimate of petroleum engineers.

1. Petroleum in large amounts is measured in barrels. The numerals 10, 20, 30, etc., on the graph represent billions of barrels. For example, the oil reserve in the United States is 30.3 billion barrels, or 30,300,000,000 barrels. Write this in words.

ESTIMATED RESERVES OF PETROLEUM



2. How many billion barrels does Saudi Arabia have in reserve?
3. Which country has 5 times as much petroleum reserve as Canada?
4. Which country has $\frac{1}{2}$ as much oil reserve as Iran?
5. The United States has how much more oil reserve than Russia?
6. What is the total petroleum reserve of the three North American countries listed?
7. Kuwait, Saudi Arabia, Iran, and Iraq are the large oil-producing countries of the Near East. What is the total oil reserve of these four countries?
8. The oil reserve of the Near East countries is how much greater than that of the North American countries?
9. Iran, Saudi Arabia, and Kuwait have more oil reserve than all the other countries shown together. How much more?

Step 5 of the problem-solving steps asks you to *Set Up and Solve the Conditional Statement(s)*. This is probably the most important of the problem-solving steps for two reasons.

a. The conditional statement reveals the mathematical relationships in the problem so you can study them without being confused by other distracting elements in the problem situation.
b. Using the methods of mathematics, you can find an equivalent statement that gives you the solution.

The following conditional statements are equations that can be solved readily if you recognize the addend-addend-sum relationship, or the factor-factor product relationship. For each equation, first write the equivalent equation that has the unknown, as represented by the variable, alone on one side. Then solve the equation.

EXAMPLES

1.

$$18N = 72 \qquad x \times y = p$$
$$N = 72 \div 18 \qquad y = p \div x$$
$$N = 4$$

2.

$$N - 19 = 34 \qquad s - a = b$$
$$N = 34 + 19 \qquad s = b + a$$
$$N = 53$$

1. $N + 5 = 40$
2. $11 + N = 34$
3. $15N = 60$
4. $N - 25 = 11$
5. $15 + 35 = N$
6. $80 = 5N$
7. $225 \div N = 15$
8. $29 - N = 16$
9. $17 + N = 40$
10. $\frac{54}{N} = 18$
11. $38 = 8N$
12. $39 = N + 11$
13. $36 \div N = 3$
14. $15 \times 13 = N$
15. $N - 15 = 25$
16. $18 = \frac{144}{N}$
17. $18 - N = 2$
18. $19 + 23 = N$
19. $5N = 45$
20. $N \div 12 = 4$
21. $14N = 70$
22. $45 - N = 28$
23. $N - 56 = 18$
24. $256 \div N = 32$
25. $N \div 5 = 15$
26. $28 - 17 = N$
27. $45N = 225$
28. $45 = N - 8$

In solving each of the following problems be sure to use all of the problem-solving steps. Set up the conditional statement for each problem. If there is a hidden question, or if there are several hidden questions, each will require a conditional statement with a different letter representing each number you are to find.

Do not be surprised if you find some data in some of the problems that you do not need to use. Problems in real life do not present themselves with the data all sorted out.

Solve each of the following, listing any unnecessary data.

1. There are 31.5 gal. in a barrel and 4 qt. in a gallon. How many gallons are there in 8 barrels?
2. A passenger plane can climb at the rate of 1500 feet per min., and has a cruising speed of 350 m.p.h. How long will it take the plane to travel 2800 miles?
3. Mildred purchased a pair of shoes at $7.49 and a pair of gloves for $1.98. How much change should she receive from a $10.00 bill?
4. When Jim started on an automobile trip, he set the odometer at zero. When he stopped for lunch it registered 129 miles. At Milltown it registered 197 miles. How far was he from Milltown when he stopped for lunch?
5. Mr. Adams has 1600 apple trees in his 3-acre orchard. Last year the orchard produced 11,200 bushels. What was the yield per tree?
6. An automobile traveled one mile on the Bonneville Flats in Utah in 9 seconds. What speed, in miles per hour, is this?
7. John is employed in a drugstore as a soda-fountain clerk at $1.15 an hour. He works six hours after school, four days a week. What does he earn each week?
8. In one hour Mike can load 40 sacks of potatoes on a wagon and haul them to a storehouse. He is paid 95¢ an hour. How many sacks can he load and store in 8 hours?
9. It is 1500 miles from Los Angeles to Chicago and 2400 miles from Los Angeles to New York. A jet plane flew from Los Angeles to New York in 4 hours. What was the average speed (miles per hour)?
10. In a recent year, the 500-mile Indianapolis Automobile Race was covered in 3 hours 40 minutes. What was the average speed for the distance?
11. John is employed at a filling station and is paid 85¢ an hour. He works 4 hours after school, five days a week. What does he earn each week?

12. Henry purchased a pair of shoes at \$7.95, 2 pairs of socks at 79¢ per pair, and a pair of gloves for \$3.59. He gave the clerk a \$10 bill and a \$5 bill. How much change should he receive?

13. Jane earns \$1.15 an hour as cashier in a store and works an 8-hour day, five days a week. If she works overtime she gets an extra 25¢ per hour. She worked 8 hours on six days last week. How much did she earn for the week?

14. Jerry earned \$50 a week last summer working in the lumber yards. He worked 8 hours a day, 5 days a week. How much per hour did he earn?

15. When Jim Brown telephones his home from college, the charge is 65¢ for the first 3 minutes and 25¢ for each additional minute. What is the charge for a 7-minute phone call?

16. Joan drove to Centerville in 3 hours at an average speed of 40 miles per hour. She used 8 gallons of gasoline. At what time must she start back if she wishes to reach home at 6:00 P.M. and if she drives at the same rate of speed?

17. The average man can walk 3 miles per hour. A horse can run a mile in about 2 minutes. How long will it take a man to walk 36 miles?

18. A plane can climb at the rate of 3000 feet per minute and has a cruising speed of 350 miles per hour. How many miles can it cruise in 4 hours?

19. When the Rosburg family started an automobile trip, Mike set the odometer at zero. When they reached Crookston, it registered 218 miles. At Watertown it registered 297 miles. How far is it from Crookston to Watertown?

20. Mr. Olsen bought a house and lot for \$15,000. He rented the house for \$100 a month for two years. He then sold the house for \$3000 more than he paid for it. How much did he sell it for?

21. A gasoline tank holds 16 gallons. Gasoline costs 35 cents per gallon. How much will it cost to fill the tank if it is $\frac{1}{4}$ full?

22. Nancy bought $1\frac{3}{4}$ yards of wool plaid for a skirt. The price was \$3.60 per yard. How much change should she receive from a \$10.00 bill?

23. Rosalie averages \$5.00 a week from her baby-sitting jobs. She allows \$3.75 a week for miscellaneous expenses and saves the rest for college. How much will she save in a year?

24. The Nelsons were taking a trip of 360 miles. Mr. Nelson drove 2 hours, averaging $52\frac{1}{2}$ miles per hour. At what average speed must he drive to finish the trip in another 5 hours? (How many hidden questions are there in this problem?)

PROPORTIONS IN PROBLEM SOLVING

A problem that can be solved by setting up a proportion can be recognized because it contains two equal ratios one of which has an unknown term. (Do you recall the unit on proportion in Chapter 6?)

EXAMPLE

An airplane flies from San Francisco to Denver, a distance of 1000 miles, in 2.5 hours. At this rate, how long will it take the plane to fly from Denver to Washington, a distance of 1500 miles?

Do you see that the ratio $\frac{2.5}{1000}$ is equal to the ratio $\frac{N}{1500}$?

So you have the proportion:

$$\frac{2.5}{1000} = \frac{N}{1500}$$

Do you remember how to solve this proportion? Try it!

Note that you can also solve the problem by first answering the hidden question: How far does the plane fly in one hour? If you try solving the problem by this method, you will see how the proportion simplifies the solution by eliminating at least one step in problem solving.

Use proportions to solve each of the following problems.

1. An automobile traveled 50 miles on 3 gallons of gasoline. How far should it travel on a full tank of 20 gallons?
2. A truck driver took 1.5 hours to drive 72 miles. At that rate, how long will it take him to drive 420 miles?
3. A painter wishes to thin his paint with turpentine by adding one pint of turpentine to 20 pints of paint. How much turpentine will he need for 5 gallons of paint?
4. Sugar is advertised at 5 pounds for 47¢. What will 25 pounds cost at that rate?
5. An automobile wheel makes 49 revolutions in traveling 375 feet. How many revolutions per mile does it make?
6. Jim earned $6.30 in 3.5 hours picking cherries in Mr. Olsen's orchard. If he continues at the same rate, how much can he earn in an 8-hour day?
7. The height of a house 40′ high is represented on a scale drawing by a segment measuring 7.5″. The garage is 16′ high. On the drawing, what is the measure of the segment representing its height?

8. Water is flowing through a pipe at the rate of 74 gallons each 5 seconds. How long will it take for 1110 gallons to flow through the pipe?

9. In 12 minutes Henry printed 175 calling cards on his printing press. How long will it take him, at this rate, to print 1400 calling cards?

10. A salesman received a commission of \$450 for selling a house for \$9000. At that rate, what would be his commission for selling a \$20,000 house?

11. The measures of the sides of a triangle are 9″, 12″, and 15″. The shortest side of a similar triangle measures 3″. What is the length of each of its other two sides?

12. Henry picked 20 bushels of apples in 3 hours. He was paid 20¢ a bushel. How much could he earn, at this rate, in an 8-hour day? (Show how you can do this problem in two ways.)

13. A Boy Scout troop hiked 10 miles in 4 hours. At that rate, how long will it take the troop to hike 17.5 miles?

14. Jim bought 6 gallons of gasoline for \$1.92. What would it cost, at that rate, to fill a 20-gallon tank?

15. The distance by air from Chicago to Kansas City is 473 miles. On a map this is represented by a segment measuring 5.5 inches. On the same map, Minneapolis and Pittsburgh are 10 inches apart. What is the distance by air from Minneapolis to Pittsburgh?

16. Mary can read 24 pages of a novel in 40 minutes. The novel is 384 pages long. How long will it take her to finish the book at this rate?

17. A steel beam 15 feet long weighs one ton. What is the weight of a similar beam that is 25 feet long?

18. An airplane is scheduled to fly from San Francisco to Salt Lake City, a distance of 700 miles, in 1.5 hours. At this rate, how long should it take to fly from San Francisco to New York City, a distance of 2975 miles?

19. An automobile wheel makes 432 revolutions in traveling 0.3 miles. How many will it make in traveling 0.7 miles?

20. The White Sox won 40 of their first 72 games. If they continue to win at the same rate, how many of their 162 games can they expect to win this season?

21. On a blueprint, $\frac{3}{8}$ of an inch represents 6 feet. How many inches will represent 32 feet?

22. A tree in the school yard casts a shadow 60 feet long. At the same time, one of the 9-foot-high posts around the tennis court casts a 13.5-foot shadow. What is the height of the tree?

1. The width of a city lot is equal to one-half of the length. The lot is 75 feet wide. How long is it?
2. A 16-foot plank is to be cut into two pieces such that the length of one piece is equal to three times that of the other. How long should each piece be?
3. A rectangle is 9″ long and 6″ wide. A similar rectangle is 6″ long. How wide is it?
4. Mike says that $\frac{1}{5}$ of his problems on his last test were wrong. He had 16 problems right. How many were wrong?
   HINT: What fraction of the problems were right?
5. Edward and Harry purchased a boat to use on the lake where they fish. Edward paid $\frac{2}{3}$ of the cost of the boat. His share was \$12. How much did Harry pay?
6. Three times a number decreased by 4 equals 20. What is the number?
7. Jim works at the filling station on Saturdays. Last week he saved \$1.50. If this is $\frac{3}{8}$ of what he earned, how much did he earn?
8. In Miss Henderson's mathematics class there are 4 more boys than girls. The enrollment in the class is 32. How many are boys?
9. Harry and his father left home at 9 A.M. to drive to Centerville, a distance of 324 miles. After $2\frac{1}{2}$ hours they stopped for lunch and found that they had traveled 120 miles. If they take half an hour for lunch and continue to drive at the same rate, at what time may they expect to arrive at Centerville?
10. Jane and Mabel together sold 66 tickets to the school play. If Jane had sold 6 more tickets, she would have sold twice as many as Mabel. How many tickets did each sell?
11. The ratio of the width of a field to its length is $\frac{3}{5}$. The width is 120 rods. How long is it?
12. A box of books weighs 37 pounds, including the weight of the box. Each book weighs $1\frac{3}{4}$ pounds. The box weighs 2 pounds. How many books are in the box?
13. A bicycle wheel makes 30 revolutions in traveling 250 feet. How many revolutions per mile does it make?
14. Mike paid \$3.39 for 10 gallons of gasoline. At that price, what will it cost to fill a 15-gallon tank?
15. Mary can type 8 pages of manuscript in 52 minutes. She agreed to type an article of 20 pages. How long will it take her to type it?

You will find that some of the information needed to solve each of these problems is missing. Study each problem and state what kind of information is lacking. Then supply the information that seems reasonable, and solve the problem.

1. A plank is to be cut into two pieces so that one piece is twice as long as the other. How long will each piece be?

2. One number is 4 times another. What are the two numbers?

3. The perimeter of a rectangle is 36 inches. How long is each side?

4. The sum of two numbers is 42. What are the two numbers?

5. Margaret and Alice together sold 18 tickets to the class play. How many did each sell?

6. The measure of one side of a triangle is 5″ greater than the shortest side. The third side measures 3″ greater than the shortest side. How long is each side?

7. George and his dog together weigh 128 pounds. How much does each weigh?

8. The two sides of equal measure of an isosceles triangle are each 4″ greater than the base. How long is each side?

9. There are 35 pupils in Miss Adams' class. How many girls and how many boys are in the class?

10. Mabel and Helen together have $10. How much does each have?

11. Mike and his father drove 120 miles on a recent trip in their car. How far did each drive?

12. The perimeter of a scalene triangle is 24 inches. How long is each side?

13. The length of a rectangular field is 20 rods greater than its width. What are its dimensions?

14. John, Henry, and Jim worked for Mr. Smith in his orchard last week. John earned three times as much as Jim, and Henry earned twice as much as Jim. How much did each earn?

15. Harry and Eric purchased an outboard motor. Harry paid $12 more than Eric. How much did each pay?

16. The perimeter of an isosceles triangle is 120 feet. How long is each side?

17. A field is twice as long as it is wide. What is its length?

18. The measure of one side of a triangle is 3′ greater than that of the second side. The measure of the third side is 6′ greater than that of the second side. How long is each side?

EXPRESSING COMPARISONS IN PER CENT

You used per cent for comparing numbers in Chapter 6. Per cent is another way of saying "per hundred." The symbol "%" means "$\times$ 0.01." Hence any ratio may be expressed as per cent if it is first expressed as a fraction whose denominator is 100.

EXAMPLE

Jim earned \$180 last summer, while Henry earned \$135. Henry's earnings are what per cent of Jim's earnings?

The ratio of Henry's earnings to Jim's earnings is, as a fraction, $\frac{135}{180}$, or in simplest form, $\frac{3}{4}$. To express $\frac{3}{4}$ as a per cent, we use the conditional statement:

$$\frac{3}{4} = \frac{n}{100}$$

$$4n = 300 \qquad\qquad x \times y = p$$

$$n = 300 \div 4 \qquad\qquad y = p \div x$$

$$n = 75$$

Then $\frac{3}{4} = \frac{75}{100}$

and $\frac{3}{4} = 75\%$

Then Henry's earnings were 75% as great as Jim's earnings.

In the exercises below, do the following:

a. First write the ratio as a fraction.
b. Set up the proportion to find the equivalent fraction whose denominator is 100.
c. Solve the proportion, using the factor-factor-product relationship.

1. Jane earned \$30 last week and saved \$6. What per cent of her earnings did she save?

2. A tire regularly priced at \$25 is sold for \$20. The reduction is what per cent of the regular price?

3. The Cleveland High School basketball team has played 15 games this year and won 9 of them. What per cent of its games has it won?

4. At a clearance sale, Mike bought a golf club regularly priced at \$12 for \$9. What per cent of the regular price was the reduction?

5. Mary is saving to purchase a bicycle that will cost \$36. She has already saved \$22.50. This is what per cent of the cost of the bicycle?

6. On a test of 20 questions, Jim had 17 correct. This is what per cent of the number of questions on the test?

7. There are 32 pupils in Mr. Jenkins' mathematics class. Of these, 20 are boys. What per cent of the class are boys?

A proportion is useful as a conditional statement in solving percentage problems. In any problem involving per cent you will find two equal ratios. One of these is the per cent written as a fraction whose denominator is 100. The second is the ratio between the other two numbers in the problem. One of the ratios will have an unknown term. By writing and solving the proportion stating that the two ratios are equal, you will find the missing term.

First, determine which of the terms is missing, and identify it by a letter. Then set up and solve the proportion.

EXAMPLES

1. Jim earns \$48 a week. He is saving 62.5% of it to buy a motorcycle. How much per week does he save?

   Let n = how much per week he saves. Also, $62.5\% = \frac{62.5}{100}$

   Then $$\frac{n}{48} = \frac{62.5}{100}$$
   $$100n = 3000 \qquad x \times y = p$$
   $$n = 30 \qquad y = p \div x$$

   Jim saves \$30 per week.

2. At the January sale, Mary bought a coat for \$32 which was 80% of its regular price. What was the regular price?
   Let n = the regular price of the coat.

   Then $$\frac{32}{n} = \frac{80}{100}$$
   $$80n = 3200 \qquad x \times y = p$$
   $$n = 40 \qquad y = p \div x$$

   The regular price of the coat was \$40.

3. Mike weighs 130 pounds. His dog weighs 48 pounds. Mike's weight is what per cent of the weight of his dog?

   Let n = the per cent. Then $n\% = \frac{n}{100}$

   $$\frac{n}{100} = \frac{130}{48}$$
   $$48n = 13{,}000 \qquad x \times y = p$$
   $$n = 270.83, \text{ or } 270.8 \qquad y = p \div x$$
   (to the nearest tenth)

   Mike's weight is 270.8% of that of his dog.

1. Jane had 85% of her problems correct on last week's test. There were 20 problems on the test. To find how many Jane had correct:

   a. State what n represents.
   b. Write the ratio of the number of correct problems to the total number of problems, with n representing the unknown.
   c. Express the per cent as a fraction whose denominator is 100.
   d. Write and solve the proportion.
   e. How many problems did Jane have correct?

2. During a sale a rug regularly priced at $240 was sold for $192. To find what per cent the reduced price was of the regular price:

   a. State what n represents.
   b. Write the per cent as a fraction, with n representing the unknown term.
   c. Write the ratio of the reduced price to the regular price.
   d. Write and solve the proportion.
   e. The reduced price is what per cent of the regular price?

3. The population of Clear Lake is about 24,000. The population of Clayton is about 30,000. To find what per cent the population of Clayton is of the population of Clear Lake:

   a. State what n represents.
   b. Write the per cent as a fraction whose denominator is 100.
   c. Write the ratio of the population of Clayton to the population of Clear Lake as a fraction.
   d. Write and solve the proportion.
   e. The population of Clayton is what per cent of the population of Clear Lake?

4. Helen is saving money to buy a camera that will cost $35. She has saved 60% of the cost. To find how much she has saved:

   a. State what n represents.
   b. Using n, write the ratio of the amount that Helen has saved to the cost of the camera.
   c. Express the per cent as a fraction whose denominator is 100.
   d. Write and solve the proportion.
   e. How much has Helen saved?

5. Fred has earned $60 taking care of lawns. This is 80% of what he needs to purchase a camera. To find what the camera will cost:

   a. State what n represents.
   b. Write and solve the proportion.
   c. How much will the camera cost?

For each of the following exercises, first write the proportion, with N representing the missing term. Then solve the proportion, indicating with letters the factor-factor-product relationship.

EXAMPLE

Solve: 18 is 120% of N Since $120\% = \frac{120}{100}$, then

$$\frac{18}{N} = \frac{120}{100}$$

$$120N = 1800 \qquad x \times y = p$$

$$N = 15 \qquad y = p \div x$$

1. 10% of N is 30
2. 115% of 46 is N
3. 36 is N% of 80
4. 15 is 30% of N
5. N is 56% of 125
6. 24 is N% of 400
7. N is 48% of 70
8. N% of 60 is 50
9. 150 is N% of 200
10. 7% of 1800 is N
11. 75% of 360 is N
12. 65% of 800 is N
13. 46% of 360 is N
14. N% of 320 is 256
15. 0.3% of 600 is N
16. N% of 56 is 42
17. 23% of N is 92
18. 60% of N is 36
19. 32 is 80% of N
20. 45 is N% of 50
21. 453 is N% of 604
22. 1.5% of 1600 is N
23. 90 is 45% of N
24. N% of 640 is 16
25. 175% of 160 is N
26. 12 is N% of 2400
27. N is 65% of 75
28. 44% of 70 is N
29. N% of 108 is 54
30. N is 22% of 138
31. 25 is N% of 70
32. 0.4% of 360 is N
33. N% of 80 is 64
34. 250% of N is 5
35. 0.7% of N is 14
36. 80% of N is 160
37. 188 is N% of 376
38. 54 is 90% of N
39. N% of 40 is 32
40. N is 56% of 450
41. 125% of 54 is N
42. 58 is N% of 14.5
43. 3.5% of N is 7
44. 15 is N% of 600
45. N is 350% of 36
46. 24 is N% of 8
47. 0.5% of 96 is N
48. N% of 16 is 64
49. 25% of N is 35
50. N is 2.3% of 120

In solving each of the following problems, be careful to follow each of the steps for solving applied problems. At Step 5, write the conditional statement, using N to represent the unknown number. Then set up and solve the proportion.

1. The Glenwood basketball team lost 4 games this year. This represents 16% of all the games the team played. How many games did the team play?
2. Of the 20 problems on a mathematics test, Jim had 90% of them correct. How many did he miss?
3. The city planned to raise $550,000 for the community fund this year. At present all but 12% of this amount has been raised. How much remains to be raised?
4. Mr. Brown purchased 160 acres of land at $225 an acre. He paid 75% of the purchase price in cash. How much cash did he pay?
5. Mr. Brown planted 20% of his 160 acres in corn, 25% in alfalfa, and the rest in vegetables. How many acres of each did he plant?
6. The bicycle shop advertises all bicycles at a reduction of 10% from the regular price. Jim saved $5.50 by purchasing a bicycle at the sale. What was the regular price of the bicycle he purchased?
7. Mr. Adams, the real estate dealer, says that the annual rent of a house should be 12% of its value. What would a house that rents for $240 a month be worth?
8. On a purchase of $8.50 Jim payed a sales tax of 34¢. The sales tax is what per cent of the purchase price?
9. Mike weighs 140 pounds. Before the next football season he hopes to weigh 154 pounds. What per cent of his present weight does he hope to gain?
10. Arnold sells papers for 5¢ each and keeps 1¢ as his pay. What per cent of the selling price is his pay?
11. A grocer sorted a 60-pound box of apples and sold 20% of them for 12¢ a pound. The rest he sold for 10¢ a pound. How much did he receive for the box of apples?
12. At the last meeting of the debating club 8% of the members were absent. If 138 members were present, what is the membership of the club?
13. An average family with an income of $5800 spends 24% of it on food. How much is spent for food?
14. Eric received $9.50 for selling newspapers. This represents 25% of the amount of his sales. How much were his sales?

15. In 1960, the population of Springdale was 23,455. In 1965, the population was 28,146. The increase in population was what per cent of the 1960 population?

16. Jane sells magazines for 25¢ each. She receives 20% of the selling price as her pay. How much will she receive for selling 120?

17. A department store reported that 15% of its expenses last year were for advertising. If $13,500 had been paid out for advertising, what were the total expenses of the store?

18. When the bus had gone 180 miles north from San Francisco, the driver said that they had covered 25% of the distance to Portland. How far is it by highway from San Francisco to Portland?

19. The Emporium advertised a $160-dinette set to sell for $120. What was the per cent of reduction?

20. Helen purchased a bicycle for $64, paying 25% in cash. How much cash did she pay?

21. Last summer Edward earned $480 and saved $300. What per cent of his earnings did he save?

22. The Jefferson High School Band is earning money to pay for new uniforms. The secretary reports that so far they have earned $690 which is 60% of what they need to pay for the uniforms. How much will the uniforms cost?

23. At a tire sale, one tire is advertised to sell for $28.50. If two are purchased, the second tire sells for only $18.50. The cost of two tires at the sale is what per cent less than the cost of two tires bought one at a time?

24. Jim saved $12 by purchasing a camera at a sale where prices were reduced 20%. How much did Jim pay for the camera?

25. Martin saved $6.50 by purchasing his set of golf clubs at a 10% reduction from the regular price. What was the regular price?

26. What is the regular price of a baseball glove that is selling for $7.04 during a "20% off" sale?

27. A jet passenger plane travels at an average speed of 550 miles an hour. A propeller-driven plane can only travel about 60% as fast. What is the speed of the propeller-driven plane?

28. During a sale the Sport Shop advertises "All Prices Reduced 15%." What should be the sale price of a golf club regularly selling for $18?

29. At a recent election 37% of the voters in Middletown failed to vote. There were 12,411 votes cast. How many failed to vote?

30. Jane earns $8 a week, on the average, by baby-sitting. She saves 40% of what she earns. She is planning to purchase a bicycle that will cost $54.40. How many weeks must she work to pay for it?

Per cent is very useful in expressing increasing and decreasing relationships in quantities. For this reason, many problems involving per cent have to do with per cent of increase and decrease. It is important to remember that unless it is otherwise stated:

The original amount is 100%.

1. An increase from \$4 to \$5 is an increase of 25%. To explain why, answer these questions:
   a. What was the original amount, before the increase?
   b. The original amount is what per cent?
   c. How much is the increase?
   d. The increase is what per cent of the original amount?
2. A decrease from \$5 to \$4 is a decrease of 20%. Use questions similar to those in Exercise 1 to explain why.
3. During a gasoline "price war," Jim found that the cost of filling the gasoline tank on his car was reduced from \$4 to \$3. This is a decrease of what per cent?
4. After the "war" ended, the cost of filling the tank rose from \$3 to \$4. An increase from \$3 to \$4 is an increase of what per cent?
5. In Exercises 3 and 4 the cost of filling the tank ended where it started. You might have expected, for that reason, that the per cents of increase and decrease should be the same. Can you explain clearly why they are different?
6. Eighty is what per cent more than 50?
   a. Since 80 is being compared to 50, 50 is __?__ %.
   b. Eighty is how much more than 50?
   c. The difference is what per cent of 50?
7. Nineteen is what per cent less than 25?
   Use questions similar to Exercise 6 to find the answer.
8. One hundred fifty decreased by what per cent is 120?
   a. What is the original amount?
   b. How much is the decrease?
   c. The decrease is what per cent of the original amount?
9. A store advertised a reduction in price of \$15 for its gas ranges formerly selling for \$74.50. What is the per cent of reduction?
10. Sixty is what per cent more than 50?
11. Twenty-five per cent less than 32 is how much?

12. Last year the enrollment in the Washington High School was 840. This year it is 945. What is the per cent of increase in enrollment? Before trying to set up the conditional statement for a problem on per cent of increase or per cent of decrease, it is useful to ask yourself two questions:

a. What is the original amount? This is 100%.

b. Is there a hidden question? If so, find the answer before setting up the conditional statement.

Here the conditional statement is: $\frac{105}{840} = \frac{n}{100}$.

Answer questions a and b, and explain how the conditional statement was set up. Then solve the proportion.

13. Last year Henry was earning \$1.20 an hour working at the filling station. This year he is selling newspapers and is able to earn only \$1.05 an hour. This is what per cent less than last year's wages?

14. At the end of the season an auto sales company advertised a reduction in price for a certain model, from \$2850 to \$2450. What was the per cent of reduction?

15. The carpenter's union secured a 25¢ per hour increase for its members. What is the per cent of increase for a member who has been earning \$3.50 an hour?

16. Seventy-two is 20% more than what number?

HINT: The percentage statement is: 72 is 120% of N, or $72 = 120\% \times N$.
What was the hidden question?

17. What number increased by 30% is 780?

18. What number decreased by 40% is 3600?

19. Fifty-six is 60% less than what number?

20. Two thousand four hundred is 20% more than what number?

21. In the United States the working time required to earn the equivalent of most basic goods has steadily decreased. The average working time necessary to earn a pound of coffee is 24 minutes. Five years ago it was 25% more than that. How much time was required 5 years ago?

22. The average worker spends 30 minutes on the job to earn the equivalent of a pound of steak. Five years ago he spent 10% more time than that. How much has this time decreased?

23. The working time necessary to purchase a movie ticket has increased. Five years ago 25 minutes working time would buy a movie ticket. Today this time has increased to 26 minutes. What is the per cent of increase?

Per Cent

A. Express each of the following as a decimal.

1. 37%
2. 60.5%
3. 0.7%
4. 14.4%
5. 37.5%
6. 179%
7. 1.03%
8. 0.08%
9. 3.2%
10. 375%
11. 111%
12. 6.05%

B. Express each of the following as a per cent.

1. 0.46
2. 7.47
3. 0.09
4. 9.5
5. 0.02
6. 8
7. 8.05
8. 0.008
9. 0.183
10. 5.02
11. 0.625
12. 0.025

C. Express each of the following as a per cent. Round to the nearest tenth of 1% if there is a remainder.

1. $\frac{3}{8}$
2. $\frac{25}{20}$
3. $\frac{4}{25}$
4. $\frac{5}{16}$
5. $\frac{13}{10}$
6. $\frac{17}{50}$
7. $\frac{6}{25}$
8. $\frac{3}{16}$
9. $\frac{10}{3}$
10. $4\frac{1}{2}$
11. $\frac{5}{9}$
12. $\frac{9}{5}$

D. Find the value of n in each of the following:

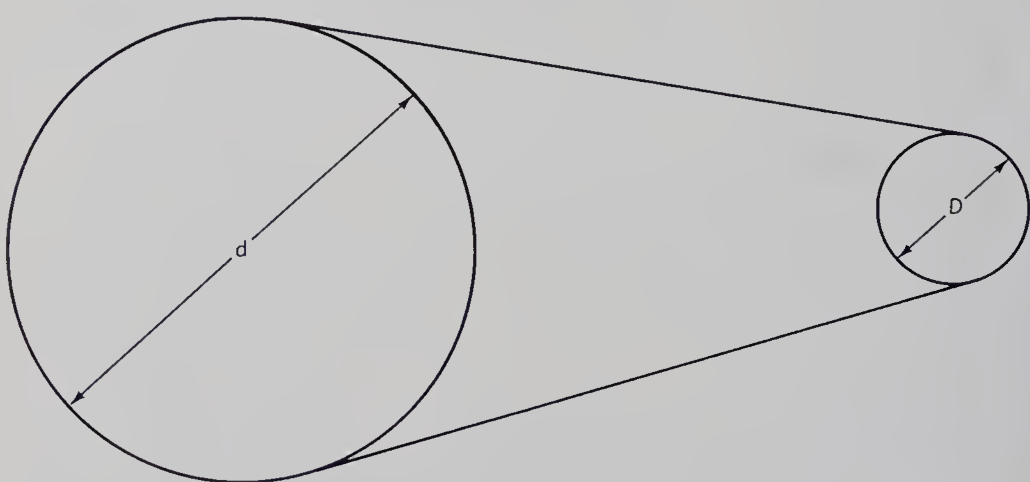
1. 30% of n is 48
2. 25% of 28 is n
3. n is 250% of 56
4. 17 is n% of 85
5. n% of 150 is 6
6. 28 is n% of 20
7. 45 is 75% of n
8. 20% of n is 8
9. 18 is 30% of n
10. n% of 4 is 16
11. 0.25% of 800 is n
12. 14.4 is n% of 20
13. n is 24% of 160
14. 3 is 25% of n
15. 45% of 80 is n
16. 50 is n% of 25
17. n% of 500 is 75
18. 30% of n is 66
19. 125% of 32 is n
20. 250% of n is 75
21. n% of 36 is 9
22. 30.2% of 40 is n
23. 24 is n% of 6
24. n is 35% of 400

If you need more practice, turn to the Practice Exercises on page 470. If not, you may work in the Experts' Corner on the following page.

Using Proportion in the Machine Shop

Do you know why the front sprocket driven by the pedals on a bicycle is larger than the one driven by the chain on the rear wheel? This is so the rear wheel will go around several times while the pedals go around only once. This makes the bicycle move more rapidly.

1. If the measure of a diameter of the front sprocket is two times as great as one of the rear sprocket, the rear wheel will turn over twice while the pedals go around once. Why?
2. If the measure of a diameter of the front sprocket on a racing bicycle is 4 times as great as one of the rear sprocket, how many times will the rear wheel turn when the pedals go around once?
3. You can compare the sizes of the front and rear sprockets by counting the number of cogs on each. Suppose there are 40 cogs on the large sprocket and 16 on the small one. How many times would the rear wheel revolve while the pedals go around once?
4. When an electric motor drives a piece of machinery, the pulley on the motor (corresponding to the sprocket with pedals) is called the *driving pulley*. The pulley turned by the belt (like the sprocket on the rear wheel) is called the *driven*. If the driven pulley has a diameter that measures three times a diameter of the driving pulley, as in the Figure below, its speed will be what fraction of that of the driving pulley?



5. The same relationship holds for the other diameters. The proportion can be written:

$$\frac{D}{d} = \frac{s}{S}$$

For this Formula, D is the measure of a diameter of the driving pulley and d of the driven. S is the speed of the driving pulley measured in revolutions per minute (r.p.m.), and s the speed of the driven, also revolutions per minute.

EXAMPLE

A motor whose pulley has a diameter that measures 4 inches (D), has a speed of 360 r.p.m. (S). It is connected to a pulley with a diameter measuring 9 inches (d). What is the r.p.m. of the driven pulley? Replacing the variables in the proportion:

$$\frac{4}{9} = \frac{s}{360}$$

or

$$1440 = 9 \times s$$

If 9 times s is 1440, what is s? What is the r.p.m. of the driven pulley?

6. The measure of the diameter of a driving pulley is 12 in. and its speed is 300 r.p.m. What is the speed of a driven pulley whose diameter measures 6 in.?
7. The measure of the diameter of a driving pulley is 3 in., of the driven pulley, 1 ft. The speed of the driven pulley is 150 r.p.m. What is the speed of the driving pulley?
8. The driving pulley on a shaft measures 2 feet in diameter. It has a speed of 100 r.p.m. What will be the speed of a driven pulley that measures 4 inches in diameter?
9. A 12 in. pulley with a speed of 200 r.p.m. is driven by a motor whose speed is 1200 r.p.m. What is the measure of a diameter of the driving pulley? Here the Formula becomes:

$$\frac{D}{12} = \frac{200}{1200}$$

Then 1200 D = 2400;
D must be $\underline{\ ?\ }$ inches.

10. A pulley measuring 4 inches in diameter is rotating at 480 r.p.m. It drives a pulley on a shaft at a speed of 960 r.p.m. What is the measure of a diameter of the driven pulley?
11. The speed of a driving pulley is 600 r.p.m., and the speed of the driven pulley is 300 r.p.m. The measure of a diameter of the driving pulley is 5 in. What is the measure of a diameter of the driven pulley?
12. A 2-inch driving pulley on a motor has a speed of 1800 r.p.m. What size driven pulley will turn at 600 r.p.m.?

Do you recall the procedures for solving general equations that were not simply expressions of addend-addend-sum or factor-factor-product relationships? Your first step is to use the additive inverse to remove variables from one side, if necessary, and constants from the other. Then if necessary you use the factor-factor-product relationship to get the variable, without a coefficient other than 1, equated to a constant. Be sure to check your solution.

EXAMPLE

Solve: $\frac{7x}{4} + 9 = x + 15$

$\frac{7x}{4} + 9 + (^-9) = x + 15 + (^-9)$	Additive inverse of 9.
$\frac{7x}{4} = x + 6$	
$\frac{7x}{4} + (^-x) = x + (^-x) + 6$	Additive inverse of x.
$\frac{3x}{4} = 6$ or $\frac{3}{4}x = 6$	$y \times x = p$
$x = 6 \div \frac{3}{4}$ or $x = 6 \times \frac{4}{3}$	$x = p \div y$ or $x = p \times \frac{1}{y}$
$x = 8$	Check: $\frac{7 \times 8}{4} + 9 = 8 + 15$

Solve each of the following equations.

1. $5x + 3 = 28$
2. $2x - 3 = 5$
3. $3x + 24 = 45$
4. $\frac{x}{2} + 5 = 9$
5. $\frac{3x}{5} - 5 = 1$
6. $2x - 17 = 8$
7. $2x + 4 = 16$
8. $4x - 8 = x + 4$
9. $2x - 9 = 11 - 2x$
10. $5x = 28 - x$
11. $5x - 15 = x + 5$
12. $4x + 2 = x + 23$
13. $6x - 4 = 4x + 2$
14. $5x + 25 = 2x + 31$
15. $15x - 2 = 10x + 18$
16. $\frac{2x}{3} - \frac{4}{3} = 5\frac{1}{3}$
17. $3\frac{1}{2}x - 7 = 7$
18. $\frac{12x}{5} + 9 = 69$
19. $\frac{4x}{3} + 5 = 11\frac{2}{3}$
20. $\frac{21x}{5} - 9 = 3\frac{3}{5}$

Oral Exercises

Frequently you will find that before you can set up the conditional statement for a problem, you need to analyze the relationships among the numbers in the problem in considerable detail. For this purpose, as you have learned, the language of algebra is very valuable.

If x represents any rational number, express:

1. 7 more than the number

2. 5 times the number

3. The number decreased by 9

4. 4 more than twice the number

5. 8 less than twice the number

6. $\frac{3}{4}$ of the number

7. $\frac{1}{2}$ of the number decreased by 6

8. 10 more than $\frac{1}{4}$ of the number

9. 6 less than 15 times the number

10. 25 more than 12 times the number

11. A field is x rods wide, and the length is 10 rods more than the width. Express its length in terms of x.

12. A triangle has sides that measure x, $x + 2$, and $x + 5$. Express its perimeter in terms of x.

13. Mary is x years old, and Helen is 7 years older. Express Helen's age in terms of x.

If x represents any rational number, tell in words what each of the following expressions means.

14. $x + 15$

15. $x - 7$

16. $2x$

17. $3x + 2$

18. $4x - 9$

19. $\frac{x}{2}$

20. $\frac{3x}{4}$

21. $\frac{x}{3} + 7$

22. $\frac{2x}{5} - 12$

23. $\frac{3x}{7} - \frac{5}{2}$

24. $x + \frac{x}{3}$

25. $2x - \frac{x}{2}$

26. $\frac{x - 3}{2}$

27. $x^2 + x$

28. $\frac{2x + 5}{3}$

29. $x^2 + 4$

30. $\frac{x^2}{4}$

31. $\frac{x^2}{x}$

32. $\frac{x^2 + 1}{3}$

33. $2x^2 - 4$

34. $5x^2 + 1$

In each of the problems below, do the following:

a. Represent the smallest of the unknowns by x or some other letter.
b. Express the other unknowns in the problem in terms of the same letter.
c. Set up an equation with these expressions.
d. Solve the equation and check the answer in the original problem.

1. The measure of each of the equal sides of an isosceles triangle is twice that of the base. Its perimeter is 30 inches. What is the measure of each side?

2. In a scalene triangle the measure of the longest side is 11 inches greater than that of the shorter. The measure of the third side is 3 times that of the shorter. The perimeter is 36 inches. What is the measure of each side?

3. If 3 times a certain number is decreased by 5 the result is the same as if twice the number were increased by 3. What is the number?

4. The length of a field is three times its width. The perimeter is 480 rods. What is the length of the field? What is the width of the field?

5. Jim and Martin together earned $75 selling subscriptions to a magazine. Jim earned one and a half times as much as Martin. How much did each of the boys earn?

6. A machinist and his helper together earn $45 per day. The helper earns $\frac{2}{3}$ as much as the machinist. How much does each earn?

7. Three times a number decreased by 2 is equal to twice the number increased by 3. What is the number?

8. In a right triangle the measure of one of the acute angles is 3 times that of the other. What is the measure of each angle?
HINT: What is the sum of the measures of the angles in a triangle?

9. The measure of the largest angle in a scalene triangle is 3 times the measure of the smallest. The measure of the third is twice that of the smallest. What is the measure of each angle?

10. The measure of one of the angles in a triangle is equal to the sum of the measures of the other two. One of the others measures 10° more than the other. What is the measure of each?

11. The measure of the largest angle in a triangle is twice that of the smallest. The measure of the third angle is 20° greater than that of the smallest. What is the measure of each?

12. In a right triangle the measures of the two acute angles are the same. What is the measure of each angle?

1. Think of a number. Multiply it by 5; add 6 to the product; multiply this sum by 4; subtract 4; divide by 20; subtract 1. You get the original number. Why? Because you followed these steps:

(1) Think of a number	(1) n
(2) Multiply by 5	(2) $5n$
(3) Add 6	(3) $5n + 6$
(4) Multiply by 4	(4) $20n + 24$
(5) Subtract 4	(5) $20n + 20$
(6) Divide by 20	(6) $n + 1$
(7) Subtract 1	(7) n

Try it on your friends — you can make variations if you can keep track of your directions.

2. Here is another based on the same idea. Tell a friend to pick a number; add 5 to it, and multiply by 3; subtract 9, and divide by 3; subtract the number he originally selected. You can tell him what his answer is without asking him any questions. Why?

3. Write down the numeral portion of your home address; double the number named and add 5; multiply the result by 50; add your age and the number of days in a year (365); subtract 615. The first two (or more) digits of the result will represent the number of your house, and the last two digits will name your age.

4. Ask a person to multiply his age by 2; add 5 to the result; multiply the result by 50; subtract the number of days in a year (365); add the change in his pocket less than 99¢; add 115 to the last sum. The first two digits of the result will name his age and the second two digits will represent the amount of change in his pocket.

5. Here is a harder one. Ask a friend to choose a number greater than 1 but less than 10. Tell him that to his number you will add a number not greater than 10. He can then add to this, a number not greater than 10. You take turns adding numbers not exceeding 10, and you can always reach 100 first even though your friend selects the first number. How does this work?

6. Ask a friend to choose a number and multiply it by 3, then tell you whether the product is an even or an odd number. If even, ask him to multiply it by $\frac{1}{2}$. If odd, have him add 1 and then multiply this sum by $\frac{1}{2}$. Now ask him to multiply the result by 3, then divide the product by 9, dropping any remainder, and tell you the quotient. Calling the quotient x, the number he started with was $2x$ or $2x + 1$, depending on whether the result of multiplying it by 3 at the start was even or odd.

You have already learned, in a previous chapter, the steps for solving mathematical problems. They are simpler than the steps for solving applied problems since in the latter, considerable attention needs to be given to identifying the mathematical relationships among the elements in the situation. In mathematical problem solving attention is directed primarily to exploration and discovery of new facts while searching for clues to the solution. This is illustrated in the following computational exercises.

STEPS FOR SOLVING MATHEMATICAL PROBLEMS

1. Understand the problem.
2. Analyze the data.
3. Discover new facts.
4. Follow up and verify promising leads.
5. Review your solution.

Each of the following puzzles with missing digits is readily solved if you look for the clues.

1. $\begin{array}{r} \text{xx} \\ \text{2x} \\ \hline \text{x43} \\ \text{5x}\phantom{\text{x}} \\ \hline \text{x8x} \end{array}$

Clue 1. $4 + x = 8$. What is the second partial product?
Clue 2. What is the multiplicand?
Clue 3. The first partial product ends in 3. What is the second digit of the multiplier?

2. $\begin{array}{r} \text{xx} \\ \text{x5} \\ \hline \text{2x5} \\ \text{xx1}\phantom{\text{x}} \\ \hline \text{xx4x} \end{array}$ **3.** $\begin{array}{r} \text{xx} \\ \text{5x} \\ \hline \text{5x} \\ \text{x4x}\phantom{\text{x}} \\ \hline \text{1x0x} \end{array}$ **4.** $\begin{array}{r} \text{xx} \\ \text{x9} \\ \hline \text{xx8} \\ \text{xx6}\phantom{\text{x}} \\ \hline \text{2x0x} \end{array}$ **5.** $\begin{array}{r} \text{xx} \\ \text{98} \\ \hline \text{xx} \\ \text{xxx}\phantom{\text{x}} \\ \hline \text{xxxx} \end{array}$ **6.** $\begin{array}{r} \text{xx} \\ \text{8x} \\ \hline \text{x7} \\ \text{2x2}\phantom{\text{x}} \\ \hline \text{xxxx} \end{array}$ **7.** $\begin{array}{r} \text{xx} \\ \text{9x} \\ \hline \text{x0x} \\ \text{2xx}\phantom{\text{x}} \\ \hline \text{xx3x} \end{array}$ **8.** $\begin{array}{r} \text{9x} \\ \text{xx} \\ \hline \text{19x} \\ \text{xxx}\phantom{\text{x}} \\ \hline \text{x0x6} \end{array}$

9. $\begin{array}{r} \text{x76} \\ \text{xx} \\ \hline \text{18xx} \\ \text{xxxx}\phantom{\text{x}} \\ \hline \text{xx920} \end{array}$ **10.** $\begin{array}{r} \text{2x9} \\ \text{xx} \\ \hline \text{x5x} \\ \text{xxxx}\phantom{\text{x}} \\ \hline \text{xxx06} \end{array}$ **11.** $\begin{array}{r} \text{31x} \\ \text{xx8} \\ \hline \text{25x0} \\ \text{xx0x}\phantom{\text{x}} \\ \text{6xx}\phantom{\text{xx}} \\ \hline \text{xxx70} \end{array}$ **12.** $\begin{array}{r} \text{294} \\ \text{8x} \\ \hline \text{23xx} \\ \text{xxxx}\phantom{\text{x}} \\ \hline \text{xxx72} \end{array}$ **13.** $\begin{array}{r} \text{x07} \\ \text{xx} \\ \hline \text{1xx3} \\ \text{x24x}\phantom{\text{x}} \\ \hline \text{xxxx3} \end{array}$

14. Can you complete a multiplication exercise and then erase several digits to make a puzzle similar to those above? Try it and see.

The solution of mathematical puzzles calls for the same skills in exploration and discovery that are required in extending our understanding of mathematical principles. The first and most important step is to *do something* with the data that are given in order to find some clues that may lead to the solution. Be sure to practice each of the problem-solving steps.

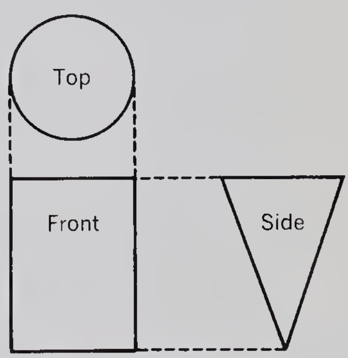
1. A dealer reduced the price of a radio from \$30 to \$24. When it still did not sell, he reduced the price to \$19.20. He finally made a third price reduction at the same per cent as the previous reductions and sold it. What was the selling price?

2. Si Perkins went to the general store to get 2 gallons of kerosene. The clerk had only an 8-gallon and a 5-gallon measure, so he said he could not measure two gallons. How could you have managed it?

3. A motorist was driving from Baldwin to Hammond, a distance of 10 miles. When he was half-way there, he found he was averaging only 30 miles an hour, so he speeded up enough to average 40 miles an hour for the total distance of 10 miles. At what rate was he traveling the last half of the trip?

4. Mary was purchasing some drawing supplies. She purchased paper at $\frac{1}{2}$¢ a sheet, pens at 1¢ each, erasers at 2 for 5¢, and pencils at 5¢ each. She found she had spent 25¢ for 25 articles. How many of each did she purchase?

5. At the right is a top view, a side view, and a front view of a familiar object. What is it?



6. A bear traveled south for three miles, then east for three miles, then north for three miles, and found himself back where he started. What color was the bear?

7. The numeral on Mr. Jensen's license plate contains five different digits. He carelessly fastened the license plate upside down on his car and found that the number that the numeral originally named was increased by 78,633. What was the numeral when right side up?

8. If a brick balances with $\frac{3}{4}$ of another brick of the same weight plus $\frac{3}{4}$ of a pound, how much does the brick weigh?

9. A strip of cloth 50 yards long is to be cut into one-yard lengths. If each cut takes one second, how long will the operation take?

10. Show how you can take $\frac{1}{2}$ of 11 and get 6.

11. A secret service officer left a counterfeit silver dollar on his desk along with 8 others that were genuine. The only way to identify the counterfeit dollar was that it weighed less than a genuine dollar. The officer arranged a balance by placing a ruler across a pencil and found the balance to be sufficiently accurate to compare the weights of the coins and to determine which one of the dollars was not genuine. Using this device he was able to spot the counterfeit coin in two weighings. This was not luck — his method assured that the counterfeit would be spotted in two weighings. What was his method?

12. A farmer was asked how many pigs he had. "Well," he said, "if I had as many more, half as many more, and $1\frac{1}{2}$ pigs more I would have two dozen." How many pigs does he have?

13. Seven paper-bound volumes of a book, each one inch thick, are in order on a shelf. A "bookworm" bores a hole straight through from the front cover of volume 1 to the back cover of volume 7. How long is the hole?

14. If 10 is the numeral that names the number of fingers on one hand, what is the numeral that names the number of quarters in $2?

15. A man and two boys crossed a river on a boat that was too small to carry all three, but it would carry either the man or the two boys. How did they manage to get across?

16. On a round trip between two towns, a bus averaged 40 miles an hour one way and 60 miles an hour on the return trip. What was the average rate for the round trip?

17. A man who had just $1 needed $1.50. He pawned his $1 bill for 75¢, and sold the pawn ticket for 75¢. Who lost on the transaction, and how much?

18. Helen went to the well with two jars, one with a capacity of 3 pints, and the other 5 pints. She measured out and brought back just 4 pints. How did she do it?

19. A shepherd was asked how many sheep he had. He said he had forgotten how many he had. He knew, though, that if he counted them by 2's, 3's, 4's, 5's or 6's there was always one sheep left over, but if he counted by 7's it came out even. What was the least number of sheep he might have?

20. Jim can mow a lawn that is 100 feet square in one hour. At the same rate, how long should it take him to mow a lawn that is 50 feet square?

21. If 5 cats can catch 5 mice in 5 minutes, how many cats are required to catch 100 mice in 100 minutes?

Part One

A. Solve each of the following proportions.

1. $\frac{x}{9} = \frac{2}{3}$
2. $\frac{9}{12} = \frac{x}{48}$
3. $\frac{3}{10} = \frac{x}{20}$
4. $\frac{5}{18} = \frac{15}{x}$
5. $\frac{5}{9} = \frac{20}{x}$
6. $\frac{6}{15} = \frac{x}{21}$
7. $\frac{x}{12} = \frac{12}{18}$
8. $\frac{4}{9} = \frac{16}{x}$
9. $\frac{5}{15} = \frac{x}{9}$
10. $\frac{6}{x} = \frac{9}{12}$
11. $\frac{15}{x} = \frac{5}{24}$
12. $\frac{15}{36} = \frac{x}{6}$

B. Find the value for N that makes each statement true.

1. 15 is $N\%$ of 25
2. $N\%$ of 35 is 14
3. 42 is $N\%$ of 56
4. 38 is $N\%$ of 19
5. $N\%$ of 240 is 3
6. 3.5% of 160 is N
7. N is 250% of 76
8. 0.3% of 160 is N
9. 125% of 124 is N
10. N is 8.2% of 70
11. 5.4 is 9% of N
12. 12 is $N\%$ of 2400
13. 16 is 0.4% of N
14. 3.5% of 96 is N
15. 6% of 1800 is N
16. 0.2% of 140 is N
17. 160% of 85 is N
18. 75% of N is 210
19. 84 is $N\%$ of 105
20. 1.7% of N is 102

C. Solve each of the following equations.

1. $7x + 3 = 5x + 13$
2. $2x + 6 = \frac{x}{2} + 13.5$
3. $\frac{2x}{3} - 10 = \frac{x}{3} - 2$
4. $\frac{2x}{3} + 1\frac{1}{3} = x - 1$
5. $\frac{3x}{4} - \frac{5}{4} = \frac{x}{4} + \frac{1}{4}$
6. $\frac{5x}{2} - 3 = 2x$
7. $\frac{5x}{2} = 7 - x$
8. $c + 12 = 68$
9. $10n = 70$
10. $b + 26 = 54$
11. $15x = 75$
12. $p - 25 = 15$

Part Two

A. Each of the equations on the left is in a form indicated by one of the factor-factor-product, or addend-addend-sum relationships on the right. List the numerals 1 through 6. After each write the letter to indicate which relationship on the right is indicated. Then write the value for x that makes the equation true.

1. $3x = 15$	**a.** $x = p \div y$
2. $x + 7 = 18$	**b.** $x \times y = p$
3. $x - 7 = 9$	**c.** $p \div x = y$
4. $18 \div x = 6$	**d.** $a + b = s$
5. $4 = 18 \div x$	**e.** $a = s - b$
6. $x = 12 - 4$	**f.** $s - a = b$

B. Write the numerals 1 through 5 on a sheet of paper. After each numeral write the answer to the corresponding question below. Refer to the following proportion.

$$\frac{7}{8} = \frac{N}{64}$$

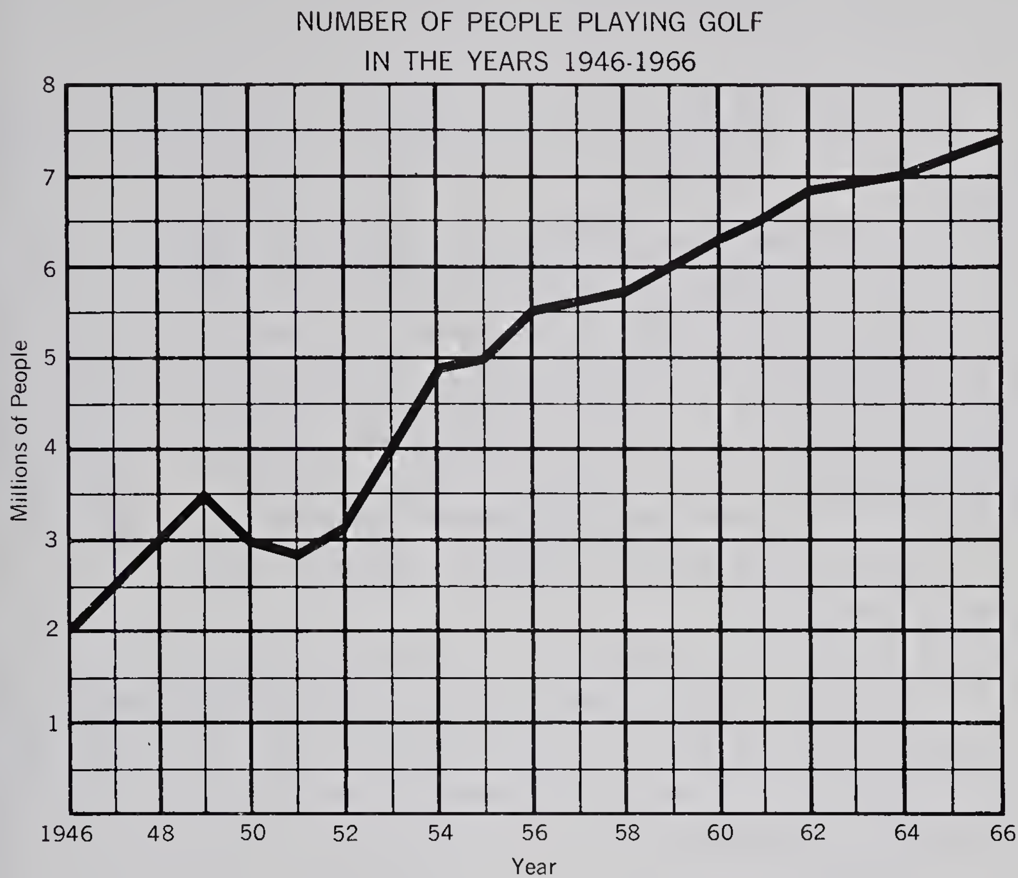
1. What are the extremes?
2. What are the means?
3. What is the product of the extremes?
4. What is the product of the means?
5. What value for N will make the statement true?

C. Set up the conditional statement and find each "certain number."

1. The sum of a certain number and 5 is 17.
2. The quotient of a certain number divided by 8 is 13.
3. The product of a certain number and 17 is 136.
4. If 17 is subtracted from a certain number, the result is 29.
5. The product of 19 and a certain number is 171.
6. If 29 is subtracted from a certain number, the result is 21.
7. If a certain number is divided by 18, the quotient is 27.
8. If 56 is added to a certain number, the sum is 39.
9. The product of 37 and a certain number is 666.
10. Five-eighths of a certain number is 25.
11. If 288 is divided by a certain number, the quotient is 12.
12. Thirty-six is $\frac{4}{9}$ of a certain number.
13. If 49 is multiplied by a certain number, the product is 686.

D. Graphs

1. The graph shows that 3 million people played golf in 1948. Approximately how many played in 1966?

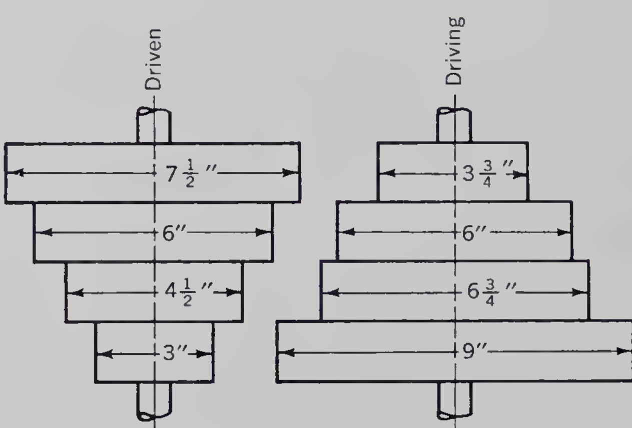


2. What was the per cent of increase in golfers from 1948 to 1966?
3. In what year did 4 million people play golf?
4. In what years did the number of golfers decrease in comparison to the previous year?
5. In 1956 about 22% of the golfers were women. To the nearest quarter million (250,000), how many women golfers were there in 1956?
6. In 1966 the per cent of women golfers was 35%. What was the per cent of increase of women golfers from 1956 to 1966?

Part Three

1. The ratio of the width to the length of a field is $\frac{5}{8}$. It is 80 rods wide. How long is it?
2. Jim, Harry, and George purchased a rowboat for \$24. Jim paid $\frac{3}{4}$ as much as Harry, and George paid \$2 more than Harry. How much did each pay?

3. A mechanic is paid \$4.50 an hour. His helper is paid 40% of that amount. How much does his helper get?
4. The population of Jamestown in 1960 was 21,245. The present population is estimated at 24,500. What is the per cent of increase?
5. Jim is saving to purchase a motorcycle. He has saved \$150 which is 37.5% of the cost. How much will it cost?
6. Jane purchased a bicycle for \$60, paying $\frac{1}{2}$ of this amount in cash and the rest in 6 equal monthly payments. How much was each payment?
7. During a sale a merchant sold a rug regularly priced at \$210 for \$150. What was the per cent of reduction?
8. A board 10 feet long is to be cut into two pieces so that one piece is 18″ longer than the other. How long will each piece be?
9. A truck driver plans to arrive at Janesville, a distance of 240 miles, in 6 hours. He covers the first 100 miles in 3 hours. At what rate must he cover the remaining distance in order to arrive on time?
10. A car travels 100 miles in 1.5 hours. At that rate, how far will it travel in 5 hours?
11. To get different speeds from a driving shaft running at a constant (never changing) speed, several pulleys giving different ratios may be fastened to each shaft. In the Figure below, the driving pulley runs at 240 r.p.m. What is the speed of the driven pulley for each of the four combinations?



12. A board 20 feet long is cut into two pieces, one 8 feet longer than the other. What is the measure of each piece?
13. Helen's father is 24 years older than Helen. He is four times as old as Helen. How old is each?
14. John, Eric, and Edward together earned \$21.50 last Saturday picking strawberries for Mr. Adams. John earned twice as much as Eric, and Eric earned 50¢ more than Edward. How much did each earn?

Part One

A. *Add:*

1. $36 + 29 + 513 + 87$
2. $\frac{3}{4} + \frac{2}{3} + \frac{5}{6} + \frac{1}{2}$
3. $6\frac{2}{5} + 8\frac{7}{10} + 4\frac{4}{15} + 5\frac{1}{2}$
4. 4.06 in. + 8.34 in. + 5.27 in. + 6.84 in.
5. \$49.75 + \$26.38 + \$37.86 + \$43.54

B. *Subtract:*

1. $243 - 68$
2. $\frac{7}{8} - \frac{1}{3}$
3. $8\frac{1}{6} - 5\frac{2}{3}$
4. \$72.14 – \$46.25
5. 36.8 ft. – 26.9 ft.
6. 36.2 lb. – 18.15 lb.

Part Two

A. Find the products.

1. 48×13
2. $\frac{5}{6} \times \frac{3}{10}$
3. $5\frac{2}{5} \times 6\frac{2}{3}$
4. $\$19.25 \times 17$
5. 0.008×3.064
6. $3\frac{3}{4} \times \frac{5}{6}$

B. Find the quotients rounded to the nearest tenth, where necessary.

1. $156 \div 12$
2. $225 \div 45$
3. $3\frac{1}{3} \div 2\frac{1}{2}$
4. $36 \div 5.2$
5. $78.14 \div .065$
6. $2\frac{5}{8} \div 1\frac{3}{4}$

Part Three

A. Write each of the following as decimals rounded to the nearest thousandth.

1. $\frac{1}{3}$
2. $\frac{2}{5}$
3. $\frac{5}{8}$
4. $\frac{7}{15}$
5. $\frac{4}{9}$
6. $\frac{9}{17}$

B. Write each of the following as a per cent. Round to the nearest tenth of 1%, where necessary.

1. $\frac{3}{5}$
2. $\frac{1}{6}$
3. $\frac{2}{3}$
4. $\frac{5}{8}$
5. $\frac{3}{13}$
6. $\frac{5}{26}$
7. $\frac{9}{5}$
8. $\frac{12}{7}$

Part Four

Write the following ratios in three ways: as a fraction in simplest form, as a decimal to the nearest thousandth, and as a per cent to the nearest tenth of 1%.

1. 6 to 24
2. 5 to 9
3. 60 to 40
4. 24 to 27
5. 35 to 50
6. 125 to 75

Part Five

Find the value of N. Calculate the per cent to the nearest tenth of 1% if there is a remainder.

1. 62.5% of 96 is N
2. N is 37% of 90
3. 0.27% of 120 is N
4. N% of 50 is 75
5. 25 is N% of 40
6. 40% of N is 16

Part Six

If x represents any rational number, tell what each of the following expressions means.

1. $2x - 8$
2. $5x - \frac{1}{3}$
3. $x + 45$
4. $\frac{2x}{3} + 9$
5. $9 - 8x$
6. $14 + \frac{x}{2}$

Solve and check each of the following equations.

7. $x + 14 = 27$
8. $\frac{x}{2} + 3 = 13$
9. $\frac{x}{12} = \frac{1}{3}$
10. $2x + 5 = 9$
11. $\frac{x}{6} = \frac{2}{3}$
12. $\frac{2x}{3} + 14 = 20$
13. $3x = 16 - x$
14. $4x + 3 = 65$
15. $5x - 7 = 2x - 1$
16. $3x + 4 = 7 + 2x$

Part Seven

A. *Add:*

1. $^{+}9 + (^{-}6)$
2. $^{-}8 + (^{-}5)$
3. $^{+}16 + (^{+}4)$
4. $^{-}17 + (^{+}8)$
5. $^{+}5 + (^{-}17)$
6. $^{-}9 + (^{+}16)$

B. *Subtract:*

1. $^{+}13 - (^{+}6)$
2. $^{+}15 - (^{-}8)$
3. $^{-}7 - (^{+}8)$
4. $^{-}9 - (^{-}10)$
5. $^{+}16 - (^{-}3)$
6. $^{-}17 - (^{+}18)$

Part Eight

1. George worked during the summer to earn money to buy a car. At the end of the summer he had earned \$120, but this was only 75% of the price of the car which he wanted. What was the price of the car?
2. On a three-day hike the local scout troop walked $20\frac{3}{4}$ miles. The first day, the troop walked $8\frac{3}{8}$ miles and on the second day the troop walked $7\frac{5}{6}$ miles. How far did the troop walk the last day?
3. Sharon works as a cashier at the supermarket. Last week she earned \$16.15. If she earns \$.95 per hour, how many hours did she work last week?
4. Last year Jack was 5 ft. 3 in. tall. This year he is 5 ft. 6 in. tall. Find, to the nearest per cent, the per cent of increase in his height during the past year.

Part Nine

Set up an equation for each of the following problems. Solve your equations and check your answers in the original problems.

1. If five times a number is decreased by 8, the result is the same as when the number is increased by 8. What is the number?
2. When a 9-ft. pole casts a shadow 5.5 ft. long, how high is a building that casts a 110-ft. shadow?
3. A car uses 18 gallons of gas in traveling 240 miles. How many gallons would it use, at the same rate, in traveling 420 miles?
4. The measure of the sides of a triangle are x, $x + 5$, and $x + 10$. The perimeter is 45 inches. What is the length of each side?
5. Mary and Jane together earned \$60. Their wages per hour were the same, but Mary worked twice as long as Jane. How much did each earn?
6. If a plane travels 1625 miles in 5 hours, how far will it travel, at the same rate, in 3 hours?
7. Henry and Jim bought a car for \$360. Henry paid twice as much as Jim. How much did each pay?
8. Mr. Jensen's orchard is four times as long as it is wide. It requires 600 rods of fencing to go around it. Give its dimensions.

CHAPTER TEN

MATHEMATICS IN BUSINESS

WORDS TO WATCH FOR

amount	*maker*	*profit*
commission	*margin*	*promissory note*
date of maturity	*markup*	*rate of interest*
discount	*operating expenses*	*retail price*
gross	*overhead*	*selling price*
interest	*principal*	*term*

The science of mathematics was developed by man in his efforts to solve the physical, social, and economic problems that he encountered. Many of these problems created the need for a number system, for computations, and in recent years for mathematics related to electronic computers. The expansion of business and industry and the increasing need for efficiency, have made the administration of merchandising and finance increasingly dependent upon mathematics. Let us examine a venture into retailing to see why this is so.

When the city of Riverside was planning a big Fourth of July celebration Henry and George decided to set up a stand to sell hot dogs and coffee. Mr. Jones owns a stand which he rented to them for $35. They could buy buns for 25¢ a dozen. Frankfurters cost 40¢ a pound. They could buy coffee for 80¢ a pound, and a pound of coffee would make about 40 cups. They figured they could make money by selling coffee at 10¢ a cup and hot dogs at 15¢ each.

1. During the day they sold 840 hot dogs. How much money did they take in on the hot dogs?

2. They bought 70 dozen buns. How much did the buns cost?

3. They bought 84 pounds of frankfurters. What was the cost of the frankfurters?
4. What was the total cost of the hot dogs?
5. The difference between *cost* and *selling price* is called the *margin.* How much was the margin on the hot dogs?
6. Harry and George bought 15 pounds of coffee and sold 600 cups at 10¢ per cup. How much was their margin on the coffee?
7. Harry and George had these additional *expenses:* cream, $1.50; mustard, $2.00; paper cups, $6.00; paper napkins, $2.00. In addition, there was $35 rent for the stand. What was the total of these additional expenses?
8. What is left from the margin after the additional expenses are paid is the *profit.* Copy and complete this statement for the business venture of Harry and George:

Total margin from hot dogs and coffee	$?
Additional expenses	?
Profit	$?

9. After their successful venture on the Fourth of July, Harry and George decided to operate their stand again on Labor Day. Although the cost of buns had gone up to 30¢ a dozen and coffee was now 90¢ a pound, they decided to sell the coffee and the hot dogs at the same price as before. They sold 420 hot dogs. How much money did they take in on the hot dogs?
10. They bought 35 dozen buns and 42 pounds of frankfurters. What did they cost? What was the margin on the hot dogs?
11. They sold 340 cups of coffee. What did they take in by selling coffee?
12. They bought 9 pounds of coffee. What did it cost?
13. What was the margin on the coffee?
14. What was their total margin for the hot dogs and the coffee?
15. Their expenses, in addition to the $35 for rent of the stand, were: paper cups, $2.50; cream, $1.00; napkins, $1.00; mustard, $1.50. What were the total additional expenses?
16. Prepare a statement, as in Exercise 8, for the Labor Day operation.
17. On the Fourth of July of the following year their expenses were: rent, $45; paper cups and napkins, $4; cream, $1.50; and mustard, $2. Costs were: 80 dozen buns at 45¢ a dozen; 15 pounds of coffee at 75¢ a pound; and 80 pounds of frankfurters at 45¢ a pound. By raising the price of the hot dogs to 20¢ each and of the coffee to 15¢ a cup, their receipts were $240. Prepare a statement and determine the profit.

Mr. Hall has a pet store in Cloverdale Heights. He is fond of animals and enjoys working with them. He knows, however, that he can remain in business only so long as the business provides the income to pay the expenses of the store and also provides him with an income for his family expenses and savings. This will be the case only if the selling price of the merchandise is sufficiently greater than the cost so that the margin covers operating expenses and leaves an adequate profit.

1. Mr. Hall purchases parakeets 24 at a time. He pays \$60 for 24 parakeets and sells them for \$4.95 each. His margin is what per cent of the selling price?

2. The operating expenses of the business are often called *overhead*. During one month Mr. Hall's overhead was as follows: rent, \$140.00; his salary (not his profit), \$300.00; electricity, \$20.00; fuel, \$25.00; advertising, \$24.00; food for pets in stock, \$12.50; miscellaneous, \$17.00. What was Mr. Hall's total operating expense (overhead) for the month?

3. The overhead is usually 30% of the total sales. Using the 30% figure, what were the total sales for the month (Exercise 2)?

4. Mr. Hall sets the selling price which is also called the *retail price*, so that he will have enough margin to pay his overhead and still have some left over. The amount left over is called *profit*. Therefore, the margin includes (1) the overhead and (2) the profit. The margin for one month was \$986.00. The overhead that month was \$723.50. What was the store's profit?

5. During July last year Mr. Hall took in \$911.40 from the sale of goods. The cost of the goods was \$483.20; the overhead that month was \$375.00.

a. What was the margin for July?
b. What was the profit for July?

6. Ted Hall worked in his father's store after school and on Saturdays. To make some money for himself, Ted bought two monkeys for \$22.50 each. He agreed to pay his father 25% of the selling price as his share of the overhead. Ted sold one monkey for \$45.00 and the other for \$43.00. What was Ted's total profit?

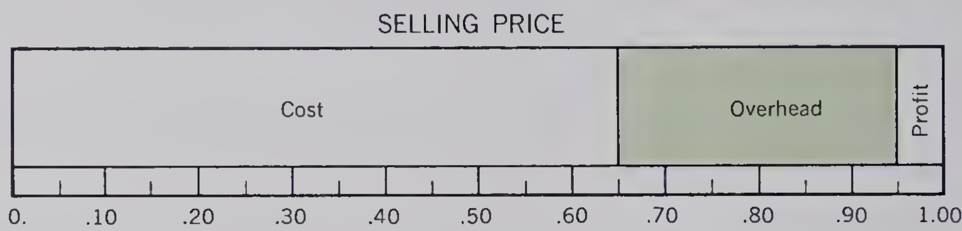
7. Mr. Hall has found that to make a profit on pets he must sell them for 50% more than they cost him. The cost of a canary trained to sing is \$7.50. He marks the selling price 50% more than the cost. What was the *markup* for the canary?

8. At what price did Mr. Hall sell the canary?

The profit, loss, and markup in business can be figured as a per cent of the cost or of the selling price. Usually the cost is considered as 100% when the per cent of markup is calculated because the merchant knows the cost at that time. When he calculates the per cent of margin or profit, however, he knows what the selling price is and can use either the selling price or the cost as 100%. Since the sales represent the amount of money available for all purposes, most merchants prefer to use the selling price as 100%. These problems will show the differences resulting from the two methods.

1. Dog food can be bought by the pet store for 10¢ a can. It sells for 15¢ a can. What is the per cent of markup figured as a per cent of the cost?
2. The pet store carries a line of dog leashes that sell for $2.25 each. They are purchased for $15.00 per dozen. Figure the margin as a per cent of (a) the cost and (b) the selling price.
3. Mr. Hall has found that 62% of all the money he takes in at the store goes to pay for the cost of the merchandise and 30% goes for the operating expenses (overhead). The remainder is profit. What per cent of the money taken in by Mr. Hall is profit?
4. The total money taken in by a business over a certain period of time is called the *gross sales*. The gross sales for the pet store during the month of September were $3000. What was the store's profit?
5. Mr. Hall must make an average profit of $200 per month (in addition to his salary) to maintain his home and family, to save money for the improvement of the store, and for his retirement. Using the per cents from Exercise 3, answer the following:
   a. What must be his average gross sales per month?
   b. What would be his average overhead if his profits were $200 per month?
   c. What is Mr. Hall's profit when his gross sales are $3750?
6. Next door to the pet store is a clothing store operated by Mr. Adams. The clothing store has an overhead of only 15% of the gross sales. The owner makes a profit of 8%. Last month the gross sales for the clothing store were $4000. How much profit did the clothing store make for that month?
7. During January the selling price of ties in Mr. Adams' store was $2.00. This price was reduced by 10% for the month of February. In March the price of ties was increased by 10%. What was the selling price of ties in February? in March?

1. In the pet store the margin on those items that are likely to spoil or go out of date is higher than on other kinds of merchandise. On some items the margin can be as high as 70% and on other items, as low as 20%. Can you think of some examples of each kind?
2. The bar graph below is divided up to show what each dollar of the selling price must pay for, on the average, from all sales for the year in the pet store. How much is the margin?



3. How much profit did Mr. Hall make on each dollar of his sales?
4. The gross sales for the year amounted to $36,000. How much was the margin?
5. How much did Mr. Hall pay for the merchandise he sold?
6. What was the profit for the year?
7. Mr. Hall allows himself a salary of $300 a month. How much is this a year?
8. What was his income per year if you add his salary to his profit?

Draw divided bar graphs (such as in the above Figure) to show these facts about three of the following. Prepare questions similar to those above about each of your graphs. Ask, for example:

a. How much of each dollar of sales goes for each purpose? (List them.)
b. Given gross sales of $40,000 a year (for example) how much is spent in each way?

9. In a clothing store the income from each dollar of sales was spent as follows: cost of merchandise, 63¢; overhead, 31¢; profit, 6¢.
10. In a confectionery store the income from each dollar of sales was spent as follows: cost of merchandise, 60¢; overhead, 33¢; profit, 7¢.
11. In a grocery store the income from each dollar of sales was divided as follows: cost of merchandise, 75¢; overhead, 19¢; profit, 6¢.
12. In the wholesale clothing business, the sales dollar was spent as follows: cost of merchandise, 80¢; overhead, 17¢; profit, 3¢.
13. In a furniture store, the sales dollar was spent as follows: cost of merchandise, 56¢; overhead, 35¢; profit, 9¢.

1. The proprietor of a shoe store pays \$8 a pair for shoes that he sells at \$12.50 a pair. He estimates that expenses amount to \$3.50 on each pair of shoes he sells. Construct a divided bar graph to show how the selling price is divided up. Label the parts S, C, O, and P for selling price, cost, overhead, and profit, respectively. Refer to the graph as you answer the following questions.
2. If you know the cost, overhead, and profit, how do you find the selling price?
3. If you know the cost, selling price, and profit, explain how you can find what per cent the overhead is of the selling price.
4. If you know the cost, margin, and overhead, how do you find what per cent the profit is of the selling price?
5. If you know the cost, selling price, and overhead, how do you find the profit?
6. Can the margin be as great as the selling price?
7. What per cent of the selling price is the margin if the cost is 100% of the selling price?
8. Can the profit be as great as the selling price?
9. Is it possible for the margin to be more than the cost?
10. If the expenses are greater than the margin, why does the merchant have a loss instead of profit?
11. If you know how many dollars the margin is, and how many dollars the selling price is, how would you find what per cent the margin is of the selling price?
12. Is the per cent of margin greater when it is figured as part of the selling price or as part of the cost?
13. If the overhead, O, is greater than the margin, M, there is a loss, L, on the transaction. The relationship can be expressed:

    $$O - M = L$$

    State this relationship in words.
14. If there is a loss on a transaction it is also true that:

    $$S = C + O - L$$

    State this relationship in words.
15. Is there always a profit if the selling price is greater than the cost? Explain your answer.
16. Can the loss be as great as the selling price? Explain your answer.
17. Explain the formula, $S = C + O + P$.
18. If C is \$56, O is \$8, and S is \$70, what is the value of P?

Mr. Henderson is the proprietor of a neighborhood grocery store. Like all merchants who handle perishable goods, he has to calculate cost and margin very carefully since the margin must cover the cost of fruits and vegetables that spoil. To clarify his calculations, Mr. Henderson uses a formula.

To show how the income from sales is used, Mr. Henderson set up the Formula $S = C + O + P$. Remember that the margin includes (1) the cost and (2) the overhead. Expressed as a formula, this is $M = C + O$. The Formula for the selling price becomes $S = M + P$.

In each of the following problems, write the formula, replace the letters by the known values, and then solve for the unknown.

1. Mr. Henderson buys coffee at 10 pounds for \$7.50 and sells it for 90¢ a pound. What is his margin per pound?

2. He purchases carrots at 13¢ a pound. His margin must be 4¢ a pound. How much does he sell them for?

3. Head lettuce costs 14¢ a head and sells for 20¢ a head. What is the margin?

4. Bananas cost 12¢ a pound. To allow for spoiling, his margin must be equal to $\frac{2}{3}$ of the cost. What does he sell them for?

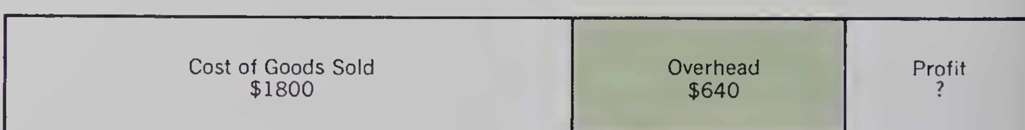
5. On tomatoes, his overhead averages $\frac{1}{5}$ of the selling price. What is his profit on a bushel of tomatoes that he purchases for \$1.40 and sells for \$2?

6. Oranges are purchased at \$4.15 per box, and are sold for \$5.50 a box. The overhead is 90¢ per box. What is the profit?

7. What rule is expressed by the formula, $S = C + P + O$? This relationship is readily seen in a divided bar graph. At the end of the week Mr. Henderson's records (to the nearest \$10) showed:

Cost of goods sold	\$1800
Total sales	\$2700
Overhead	\$ 640

The divided bar graph he drew looked like this:



How much was Mr. Henderson's profit?

8. The Total sales does not appear on the graph. What items does it include?

9. What two items does the margin include?

10. How many dollars was the margin?

11. The margin was what per cent of the cost?

12. The profit was what per cent of the cost?

13. The following week Mr. Henderson's records (to the nearest hundred dollars) were as follows:

Cost of goods sold	\$1500
Total sales	\$2300
Overhead	\$ 600

What was the profit for the second week?

14. Construct a divided bar graph to show how the income from sales was divided.

15. Lemons are purchased at 5¢ a pound and sold for 8¢ a pound. The expenses average $\frac{1}{6}$ of the selling price. How much per pound is the profit?

16. Mr. Henderson purchased 200 lb. of celery at 15¢ a pound. He sold 150 lb. at 25¢ a pound, but the rest spoiled. How much was his margin?

17. Mr. Henderson bought 300 lb. of potatoes. He sold all of them except 40 lb. which spoiled. What fractional part was spoiled?

18. If the potatoes cost \$3.60 per 100 lb., what did he pay for 300 lb. of potatoes?

19. How much per lb. did the remaining potatoes sell for if Mr. Henderson's margin was 20% of the cost of the 300 lb.?

20. Mr. Henderson bought five 40-lb. boxes of peaches at \$4.20 per box. He estimated spoilage loss of $\frac{1}{5}$ of the peaches. How much per pound should he charge so that the margin equals $\frac{1}{3}$ of the cost?

21. Mr. Henderson purchased 400 boxes of strawberries at 18¢ a box. He expects that 15% of them will spoil before selling. What must the remainder sell for per box to provide a margin of \$13 on the transaction?

SPECIAL PROJECT

Arrange for a trip behind the scenes of a local food market. Be prepared to ask questions that will show you how prices are determined. Some sample questions might be:

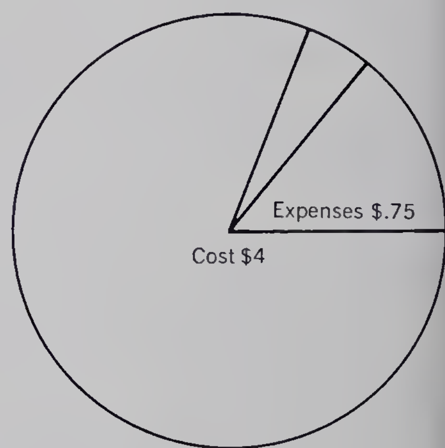
1. What per cent of the fresh fruit and vegetables bought is lost due to spoilage and cannot be sold?
2. What is the main source of supply of the fruits and vegetables sold?
3. What per cent of margin does the proprietor expect?

There are many questions you might ask. Prepare your list ahead of time to obtain all possible information.

1. Here is a list of articles sold in the Campus Book Store:

Article	*Cost*	*Selling Price*	*Margin*	*Expenses*	*Profit or Loss*
Book	$4.00	$5.00	?	$ .75	?
Stationery	.80	1.50	?	.30	?
Sport shoes	8.10	10.50	?	3.05	?
Tablet	.25	.50	?	.12	?
Stapler	1.84	2.05	?	.21	?
Sport cap	1.25	2.50	?	.42	?
Class pin	3.57	6.65	?	1.85	?
Pencil	.03	.10	?	.06	?
Notebook	.12	.20	?	.12	?

Copy the list; calculate the margin for each article; then calculate the profit or loss on each. In the *Profit or Loss* column, indicate each loss by a negative number.



2. The circle graph at the right shows the data on the book sold in the Campus Book Store. What part of the circle represents the selling price?

3. In the circle graph, which part should be labeled profit?

4. Which parts of the circle graph represent the margin?

5. In the circle graph, how many degrees represent:

a. The selling price?

HINT: Set up proportions.

b. The cost? HINT: $\frac{4}{5} = \frac{N}{360}$

c. The expenses?

d. The profit?

e. The margin?

6. Make a circle graph to show how the selling price of the sport cap is divided. Label the parts, and give the graph a suitable title.

7. Make a divided bar graph to show how the selling price of a tablet is divided, labeling the parts, and giving it a suitable title.

8. Which of the two graphs, bar or circle, do you think shows the facts more clearly? Explain why you think so.

A. Express each of the following as a decimal.

1. 14% **4.** 0.7% **7.** 3.5%
2. 31.5% **5.** 7.25% **8.** 250%
3. 0.2% **6.** 127% **9.** 0.04%

B. Express each of the following as a per cent.

1. 0.46 **4.** 0.025 **7.** 4.0
2. 1.48 **5.** 0.07 **8.** 0.003
3. 0.079 **6.** 2.5 **9.** 1.37

C. Express each of the following as a per cent, rounded to the nearest tenth of 1%.

1. $\frac{11}{25}$ **4.** $\frac{1}{200}$ **7.** $\frac{12}{7}$
2. $2\frac{3}{4}$ **5.** $\frac{60}{11}$ **8.** $\frac{3}{1000}$
3. $\frac{1}{125}$ **6.** $\frac{3}{8}$ **9.** $\frac{5}{6}$

D. Find the value for N to make each statement true.

1. 15% of 70 is N
2. 3 is N% of 50
3. 1.8% of N is 9
4. 12% of 60 is N
5. 12.5% of N is 15
6. $\frac{3}{4}$% of 480 is N
7. 1.25 is N% of 5
8. 125% of N is 60
9. 25% of 50 is N
10. 12 is N% of 96
11. 400 is N% of 225
12. 45 is 60% of N
13. 7 is 175% of N
14. 8.5 is 17% of N

E. Find the value for n in each of the following:

1. $\frac{n}{6} = \frac{1}{3}$
2. $\frac{3}{8} = \frac{n}{16}$
3. $\frac{5}{9} = \frac{10}{n}$
4. $\frac{n}{100} = \frac{5}{20}$
5. $\frac{3}{n} = \frac{75}{100}$
6. $\frac{3}{16} = \frac{n}{100}$
7. $\frac{5}{6} = \frac{20}{n}$
8. $\frac{n}{100} = \frac{4}{32}$
9. $\frac{7}{n} = \frac{21}{35}$
10. $\frac{35}{100} = \frac{n}{20}$

If you need more practice use the Practice Exercises on page 473. If not, you may work in the Experts' Corner on the following page.

Marking Merchandise in Code

Some merchants write the cost on each article so that if they want to reduce the selling price later on they can still allow for a margin. In order to avoid confusion with the selling price, a *code*, with letters representing numerals, is used for this purpose. Mr. Hall uses this code:

I	T	C	A	N	B	E	Y	O	U
1	2	3	4	5	6	7	8	9	0

Thus, an article that cost 68¢ would be labeled BY; an article that cost $2.75 would be labeled TEN. Explain.

1. What is the cost of each of the articles labeled with these letters?

a. EO **c.** IAB **e.** AYN **g.** COT
b. CE **d.** OB **f.** ICE **h.** ACE

2. What would you label each article to show these cost prices?

a. 47¢ **c.** 38¢ **e.** $8.31 **g.** $1.63
b. 89¢ **d.** $2.75 **f.** $3.46 **h.** $4.22

3. An article labeled TCU is sold for $3.60. How much is the margin?

4. In Exercise 3, the margin is what per cent of the selling price?

5. A bird bath in the pet shop was labeled CAN. How much did it cost?

6. The selling price of the bird bath during the summer was $4.60, but at the end of the summer it was sold at a 10% reduction. How much was the margin on the bird bath?

7. A dog harness labeled TCB is priced to sell for $4. The margin is what per cent of the selling price?

8. Mr. Hall had a clearance sale and reduced the selling price of the harness in Exercise 7 by 20%. How much was the margin?

9. The margin in Exercise 8 is what per cent of the selling price?

10. A druggist named A. T. Marelius used his name as a code:

A	T	M	A	R	E	L	I	U	S
1	2	3	4	5	6	7	8	9	0

To repeat a digit he used X. That is, an article that cost $2.00 would be labeled TSX. How would he label an article that cost $5.55? $5.45?

11. Mr. Marelius sold a box of soap for $5.00 that was labeled MRS. How much was his margin?

12. The margin on the soap was what per cent of the selling price?

As we have mentioned earlier, the merchant must establish a selling price for each kind of merchandise so that the margin will provide for the overhead and profit. The records of the business must be complete in order to show what per cent of the sales are needed for this purpose. This determines what the margin should be.

EXAMPLE

Mr. Hall purchased a number of bird cages for $3.60 each. The margin on these articles must be 40% of the selling price. What is the lowest price each cage can sell for?

Since the margin must be 40% of the selling price, the cost is 60% of the selling price. Why?

Then the conditional statement is:

$$3.60 \text{ is } 60\% \text{ of } N$$

The proportion is:

$$\frac{60}{100} = \frac{3.60}{N}$$

Solve the proportion.

In working the following problems, first set up the conditional statement, then the proportion, and calculate the selling price.

1. Mr. Hall purchased a supply of dog beds for $30 a dozen. The margin must be 50% of the selling price.
2. Mr. Brown purchases women's coats at $21.60 each. His margin must be 40% of the selling price.
3. The margin on a certain quality of gloves must be 45% of the selling price. The buyer purchases a lot for $13.20 per dozen.
4. The supermarket purchases oranges at 6¢ a pound and sells them so as to have a margin of 40% of the selling price.
5. The supermarket sells butter so as to have a margin of 20% of the selling price. If the cost is 48¢ per pound, what is the selling price per pound?
6. Strawberries are purchased at $51.84 for 24 dozen boxes. They are priced in order to have a margin of 40% of the selling price. What is the selling price per box?
7. The book shop purchases fountain pens at $147 per 100. The margin is to be 30%. What should each pen sell for?
8. The supermarket purchases grapefruit at $4.50 per 100 lb. The margin is to be 40%. What is the selling price per pound?

1. The earnings of a salesman, figured as a per cent of the money received for the goods he sold, are called *commissions.* Martin sells papers for 10 cents each. His commission is 20% of his sales. How much did he earn on each paper? How much does he earn by selling 100 papers?
2. Helen works in a toy store during the Christmas holidays. She receives a 10% commission on all sales. The first Saturday that she worked she sold \$120 worth of toys. How much did she earn that day?
3. Joe's father sells household appliances and receives a commission of 20% on all sales. One week he sold \$750 worth of merchandise. What was his commission?
4. Jack has a summer job in a men's store. He receives a commission of 12%. How much must he sell each week to earn \$100?
5. Jim sold magazine subscriptions totaling \$75. His commission was \$6. What per cent or rate of commission did he receive?

   HINT: The per cent is: $\frac{\text{amount of commission}}{\text{amount of sales}} = \frac{6}{75} = \underline{\ ?\ }\%$
6. The Blake Realty Company sold a house for Mr. Adams for \$9000. The company's commission was \$450. What per cent of commission did the company receive?
7. Mr. Henderson purchases supplies for a big hotel on a commission. One week he bought \$140 worth of eggs and received \$21 commission. What rate of commission is he paid?
8. Bert sold 120 bushels of apples for his father at \$3.50 per bushel. His commission was \$21. What rate of commission does he get?
9. George sold \$350 worth of vegetables and received \$28 commission. What rate of commission did he get?
10. The Ajax Company sells wheat for Mr. Adams on commission. One season his wheat sold for \$4500. The company received \$67.50 as commission. What rate of commission did the company receive?

QUESTIONS FOR DISCUSSION

1. Why do many people prefer to work on a commission rather than on a salary?
2. What are the advantages and disadvantages of working on a commission? If a dealer offered you either \$10 or a 10% commission for selling some old furniture, how would you decide which you would take?

Retail merchants occasionally have sales during which the prices on merchandise are reduced. The purpose may be to attract new customers, to make room for new merchandise, or to dispose of merchandise that is going out of date or out of season. The reduction of price is called a *discount*. The discount may be stated in cash or as a per cent of the regular price.

1. The Eagle Drug Store advertises fountain pens, regularly priced at \$4, on sale for \$2.40. How much is the discount?

2. The discount on the fountain pens is what per cent of the regular price?

3. Golf clubs are on sale at the Campus Shop for \$64 a set. The regular price is \$80. The discount is what per cent of the regular price?

4. Here are some bargains advertised at the Spring Sale at the Emporium. Copy the list, and for each item calculate the discount and what per cent the discount is of the regular price.

Article	*Regular Price*	*Sale Price*	*Discount*	*Per Cent Discount*
Sweater	\$18	\$14.40	?	?
Shoes	\$12	\$ 9.00	?	?
Watch	\$30	\$26.25	?	?
Pocketknife	\$ 4	\$ 3.20	?	?

5. A hardware store advertises all merchandise at a discount of 20%.

a. What is the sale price of an electric heater marked \$30?

b. What will a washing machine sell for if it is marked \$120?

c. Find the discount on a refrigerator that was marked \$300.

6. A bookstore allows a 10% discount on slightly soiled books. Find the sale price of a book that was marked \$4.50 when new.

7. How much does one save by buying a book regularly priced at \$5 if there is a 10% discount?

8. The same bookstore is selling secondhand (used) books at a discount of 40%. A history book cost \$6 when new. How much does it sell for secondhand?

9. A furniture store gives a 10% discount to all customers who pay cash. Mr. Adams paid cash for a chair marked \$24.50. How much did he pay?

10. During a "clearance sale" all merchandise at the hardware store was sold at a discount of 10% of the regular price. Jim saved \$2.95 on an electric drill. What was the regular price?

Oral Exercises

In working these problems without pencil and paper you may find the following list of equivalents useful. Refer to the list when necessary.

$\frac{1}{20} = 5\%$	$\frac{1}{4} = 25\%$	$\frac{1}{2} = 50\%$
$\frac{1}{10} = 10\%$	$\frac{3}{10} = 30\%$	$\frac{3}{5} = 60\%$
$\frac{1}{5} = 20\%$	$\frac{2}{5} = 40\%$	$\frac{3}{4} = 75\%$

1. Mr. Hall purchased poodle puppies for \$50 each. He sold them for \$70 each. How much was his margin on each puppy?
2. A grocer purchases eggs at 45¢ a dozen. He sells them so as to have a margin of 20¢ per dozen. What is the selling price?
3. The grocer purchases potatoes at \$2 per bushel. He sells them at \$2.75 per bushel. He figures that his expenses amount to 50¢ per bushel. How much is his profit on a bushel of potatoes?
4. A merchant's margin on a davenport that sold for \$200 was 25% of the selling price. What was the cost of the davenport?
5. A TV set that costs \$140 sells for \$200. The margin is what per cent of the selling price?
6. A typing table sells for \$15. The margin is 40% of the selling price. What was the cost of the typing table?
7. Martin sells newspapers, receiving 20% of the selling price as his commission. Last week his sales were \$50. How much was his commission?
8. A salesman in a furniture store receives 10% of his sales as commission. What was his commission on a sale of \$240?
9. A sport coat regularly priced at \$30 is sold for \$24. The discount is what per cent of the regular price?
10. A hardware store is advertising a sale at which all merchandise is sold at a reduction of 20% off the regular price. The regular price of an electric drill is \$40. What is the sale price?
11. A real estate dealer who receives 5% commission on his sales sells a house for \$24,000. How much was his commission on the sale?
12. A 35-millimeter camera regularly priced at \$60 is sold for \$48. The reduction is what per cent of the regular price?
13. Helen sells magazine subscriptions on a 10% commission. Last week she sold \$25 worth of subscriptions. How much did she receive as commission?

14. A grocer sells potatoes for \$3 a bushel. His margin is 20% of the selling price. What does he pay for a bushel of potatoes?
15. During a sale, a pair of shoes regularly priced at \$10 is reduced 20%. What is the sale price?
16. Last year the enrollment of Lincoln High School was 800. This year the enrollment has increased by 10%. What is this year's enrollment?
17. A tire regularly selling for \$30 is advertised to sell for \$24. The reduction is what per cent of the regular price?
18. A furniture store marks its merchandise in order to have a margin of 25% of the selling price. What is the margin on a rug that sells for \$120?
19. A salesman receives a commission of 20% of his sales. How much does he earn on a sale amounting to \$250?
20. Mr. Emerson says that his margin on a radio set that cost him \$80 is 30% of the cost. How much is his margin?
21. What is the selling price of the radio set that cost Mr. Emerson \$80?
22. Mr. Emerson's operating expenses are 25% of his costs. How much would be the profit on the radio set that cost Mr. Emerson \$80?
23. There are 1200 pupils enrolled in Central High School. On Thursday 4% of them were absent. How many were present?
24. Mr. Evans sold 1500 bushels of wheat. He says that this is 75% of his crop. How many bushels did he raise?
25. Last summer George was able to save \$250. He says that this was 50% of what he earned. How much did he earn?
26. A grocer purchased eggs at 36¢ a dozen, and sold them at 45¢ a dozen. His margin is what per cent of the selling price?
27. A refrigerator priced at \$250 was sold for \$225. The discount was what per cent of the regular price?
28. Henry sells newspapers on commission. Last week his sales amounted to \$60, and his commission was \$15. What per cent of his sales was his commission?
29. A sport jacket regularly priced at \$32 was sold at a discount of 25%. What was the selling price?
30. Mabel worked at the supermarket last summer, and was able to save 60% of what she earned. She earned \$500. How much did she save?
31. Mr. Adams harvested 5000 bushels of wheat last summer. He sold 80% of it at the elevator as soon as it was harvested, and stored the rest in his granary. How much did he put in his granary?

The following set of problems will show how carefully you are following the steps in problem solving.

STEPS FOR SOLVING APPLIED PROBLEMS

1. Understand the problem.
2. Note what the problem asks for.
3. Look for hidden questions.
4. Estimate a reasonable answer.
5. Set up and solve the conditional statement(s).
6. Check your answer.

1. Mr. Smith earns \$475 a month. He says that he pays out 20% of his income for house rent. What rent per month does he pay?

2. During a sale Henry bought a suit of clothes that was marked \$50 for \$36. What per cent of the regular price did he save?

3. The overhead in a grocery store averages 29% of the sales. If the sales were \$320,000, how much would you expect the expenses to be?

4. Mr. Henderson buys eggs for 46¢ a dozen and sells them for 60¢ a dozen. What per cent of the selling price is the margin?

5. By selling a suit of clothes for \$55, a merchant had a margin of 30% of the selling price. How much was the margin?

6. The rent on Mr. Henderson's fruit and vegetable stand is \$120 per month. This is 60% of his overhead. How much per month is his overhead?

7. Mr. Osborne earned \$6400 last year. He expects his earnings this year to be 15% more. How much does he expect to earn this year?

8. The Central Elevators employed an agent to purchase 4000 bu. of wheat at \$2.20 a bushel. The agent charged 2% of the cost of the wheat as commission. How much was the agent's commission?

9. Mr. Sanders purchased a radio for \$24. He set his selling price so as to have a margin of 40% of the selling price. How much was the selling price?

10. Mr. Henderson's margin on tomatoes is 40% of the selling price. His margin is 4¢ a pound. What do they sell for per pound?

11. The Campus Book Shop purchased a book for \$2.40. It was marked to sell for \$4. At the end of the school year, it was sold at a reduction of 10% from the marked price. What per cent of the selling price was the margin?

FINDING THE HIDDEN QUESTION

In each of the following problems there is at least one hidden question that must be answered before you can find the answer to the problem itself. In each problem, first state and answer the hidden question or questions. Then solve the problem.

1. Jim bought a pair of shoes for \$6.50 and a pair of rubbers for \$2.75. He gave the clerk a \$10 bill. How much change should he receive?
2. Jane is on the refreshment committee for the class picnic. Thirty-six pupils plan to attend, and ice cream is to be served. It will cost 30¢ a quart, and one quart will serve 6 pupils. How much will the ice cream cost?
3. A grocer had 80 bushels of potatoes in stock. He sold 30% of them at \$1 a bushel, and the rest at \$1.25 a bushel. How much did he receive for the 80 bushels?
4. During a sale a brief case marked \$12 is reduced 20% in price. The dealer still has a margin of 25% of the reduced selling price. How much did the brief case cost him?
5. The Acme Electric Company sells radios for \$40. The margin is 30%, and the profit is equal to $\frac{1}{6}$ of the margin. How much is the profit?
6. John and Edward bought a secondhand car. John paid $\frac{3}{5}$ of the price, and Edward paid the rest. John's share was \$180. What was the total price of the car?
7. Last month the total sales of the Acme Electric Company amounted to \$24,000 (rounded to the nearest \$100). The cost of the merchandise sold was \$18,000; overhead was \$4800. The profit was what per cent of the sales?
8. The Campus Clothing Store is having a clearance sale, with all prices reduced 20%. Jim saved \$9 by purchasing a suit during the sale. What did he pay for the suit?
9. Ted got a job painting house numerals on the curbs. He received 40% of the amount charged. If he earned \$12 after painting numerals for 30 houses, what was the charge to each customer?
10. The supermarket purchased 200 pounds of coffee at 48¢ a pound and sold it so as to have a margin of 25% of the selling price. How much was the margin on the 200 pounds of coffee?
11. By selling an overcoat for \$60, a merchant has a margin of 25% of the selling price. What did the coat cost him?
12. By purchasing a book at a sale at "20% off," Elizabeth saved 30¢. How much did she pay for the book?

13. a. Henry's father owns an orchard. Henry gets a 15% commission on all the fruit he sells. Last week Henry sold 250 lb. of pears at 16¢ per pound. How much was his commission?

b. How much commission will Henry receive for selling 320 lb. of cherries at 25¢ per lb.?

14. A rug regularly priced at $375 is sold for $225. What is the per cent of discount?

15. a. Mr. Hall purchases cat food at $2.88 per case of 24 cans. He sells it at 16¢ a can. The margin is what per cent of the selling price?

b. Mr. Hall estimates his expenses for cat food at 72¢ a case. The profit is what per cent of the selling price?

16. A dealer sold a radio for $80. His expenses were 25% of the selling price. His profit was 5% of the selling price. How much did he pay for the radio?

17. In selling shoes at $12 a pair, a dealer has a margin of $2.40. The margin is what per cent of the cost of the shoes?

18. Last November Mr. Hall's gross sales amounted to $2080. The margin was 30% of the sales.

a. What was the cost of merchandise sold?

b. In November, Mr. Hall's expenses amounted to 21% of the sales. How much was the profit?

c. Mr. Hall's profit for November was what per cent of the gross sales?

19. a. At football games, Mike Adams sold hot dogs for 20¢ each. One week his sales amounted to $40. The frankfurters and buns cost him $27, the mustard $1, and the paper wrappings 65¢. How much was his margin?

b. Mike paid the manager of the ball park $5 for the hot dog concession. He also had to pay $1.50 for delivery of his merchandise to the park. What was his profit for the week?

c. Mike's profit was what per cent of his sales?

20. Bill's dad bought $\frac{1}{2}$ dozen golf balls for $4.98, a package of tees for 35¢, and a pair of golf shoes for $12.98 at the Suburban Sport Shop. How much change will he receive from a $20-bill?

21. If a salesman at the sport shop earns a commission of 15% on all sales, how much commission will the salesman who sold the articles listed in Exercise 20 receive?

22. A dealer sold a television set for $400. His margin was 30% of the selling price. His profit was 6% of the selling price. How much were his expenses?

In each of these transactions there is at least one mistake — sometimes more. See if you can find the mistakes without pencil and paper. Then check your answer.

1.

2 lb. butter at 65¢ per lb.	$1.30
6 cans of tomatoes at 2 for 33¢	.79
5 lb. sugar at 9½¢ per pound	.48
1 lb. raisins at 34¢	.34
	$3.91

2. Mildred made these purchases:

3 lb. coffee at 79¢ per lb.	$2.37
1½ lb. butter at 66¢ per lb.	.99

She received $1.44 change from a $5.00 bill, and told the clerk he had made a mistake. How much change should she receive?

3. Jane checked the sales slip and told the clerk he had made a mistake on this one. Was she right?

3 cans of grapefruit juice at 33¢	$ .99
8 bars of soap at 4½¢ each	.40
9 cans of evaporated milk at 3 for 46¢	1.38
	$2.34

4. Henry purchased these items:

8 lb. ham at 88¢ per pound
4 lb. bacon at 86¢ per pound
4 lb. bananas at 23¢ per pound
25 lb. sugar at 9¢ per pound

He received $5.35 change from two $10 bills. Was this correct?

5. How much change should Henry receive?

6. Jim purchased an order of meat amounting to $3.17. He gave the clerk a $5.00 bill and received in change a $1 bill, a half dollar, a quarter, two nickels, and three pennies. What change should he receive, and how much more or less should it be?

7. Mike recorded these figures on the sales slip he made out. Find his mistake(s).

5 lb. sugar at 12¢ per lb.	$ .60
8 bars of soap at 2 for 7¢	.56
2 dozen eggs at 39¢ per dozen	.88
	$2.04

A. *Add:*

1. $\begin{array}{r} \$\ \ 57.75 \\ 106.45 \\ 7.95 \\ 415.56 \\ 87.50 \\ 1500.00 \\ \hline \end{array}$

2. $\begin{array}{r} \$\ 98.09 \\ 2.07 \\ 108.45 \\ 59.05 \\ 413.87 \\ 25.72 \\ \hline \end{array}$

3. $\begin{array}{r} \$200.50 \\ 35.09 \\ 115.09 \\ 700.00 \\ 56.63 \\ 16.56 \\ \hline \end{array}$

4. $\begin{array}{r} \$\ 75.00 \\ 3.05 \\ 245.05 \\ 557.45 \\ 90.80 \\ 803.03 \\ \hline \end{array}$

B. *Subtract:*

1. $\begin{array}{r} 135 \\ 90 \\ \hline \end{array}$
2. $\begin{array}{r} 415 \\ 85 \\ \hline \end{array}$
3. $\begin{array}{r} 625 \\ 156 \\ \hline \end{array}$
4. $\begin{array}{r} 113 \\ 34 \\ \hline \end{array}$

C. *Multiply:*

1. $\begin{array}{r} 305 \\ 18 \\ \hline \end{array}$
2. $\begin{array}{r} 702 \\ 67 \\ \hline \end{array}$
3. $\begin{array}{r} 450 \\ 203 \\ \hline \end{array}$
4. $\begin{array}{r} 320 \\ 155 \\ \hline \end{array}$

D. *Divide:* Round to the nearest hundredth or to the nearest cent.

1. $2284 \div 37$
2. $8272 \div 47$
3. $1470 \div 32$
4. $\$55.50 \div 56$
5. $\$98.30 \div 24$
6. $\$86.58 \div 42$

E. Find the value of each of the following:

1. $3\frac{1}{2} + ({}^{+}5\frac{2}{3})$
2. $6\frac{2}{5} + ({}^{-}8\frac{1}{10})$
3. $8\frac{3}{4} - ({}^{+}6\frac{1}{2})$
4. $4\frac{3}{8} - ({}^{-}6\frac{3}{4})$
5. $7\frac{1}{3} - ({}^{+}8\frac{1}{2})$
6. $4\frac{3}{4} + ({}^{-}5\frac{5}{8})$

F. *Subtract:*

1. ${}^{+}8 - ({}^{+}2)$
2. ${}^{-}16 - ({}^{-}4)$
3. ${}^{+}2.2 - ({}^{-}3.7)$
4. ${}^{+}9.5 - ({}^{-}8.7)$
5. ${}^{-}16 - ({}^{-}8\frac{2}{3})$
6. ${}^{+}6\frac{1}{6} - ({}^{-}3\frac{5}{6})$
7. ${}^{-}24 - ({}^{+}38\frac{5}{9})$
8. ${}^{-}18.3 - ({}^{+}9.2)$

G. Solve each of the following for n.

1. $n + 16 = 23$
2. $5n = 35$
3. $13 - n = 9$
4. $n \div 15 = 7$
5. $24 \div n = 3$
6. $n - 18 = 19$
7. $\frac{n}{7} = 6$
8. $27 = n + 5$

If you need more practice, turn to the Practice Exercises on page 477 and the following. If not, work in the Experts' Corner on the following page.

Multiplying "In Your Head"

1. Suppose you had learned the multiplication tables only through 5×5. You could use your fingers to find the product of two factors less than 11 and greater than 5, such as 9×7. Represent 9 on your left hand by extending four fingers (the excess of 9 over 5). Represent 7 on your right hand by extending two fingers. Why? The sum of the extended fingers on both hands gives you the number of tens in the product. The product of the numbers of folded fingers on the two hands gives the number of units in the product. What is the product?

2. Use your fingers to find the following products.

a. 9×6	**c.** 9×8	**e.** 7×7
b. 8×8	**d.** 7×6	**f.** 10×10

3. Do you recall how to find the product of two numbers if the numerals naming the numbers have the same tens digit, and the sum of the numbers named by the units digits is 10? (See page 37.) Using this rule, write the following products without written calculation.

a. 53×57	**c.** 64×66	**e.** 42×48
b. 72×78	**d.** 87×83	**f.** 91×99

4. A useful special case for the rule referred to in Exercise 3 is to square a number ending in 5. Examine the following products and state the rule.

a. $15^2 = 225$ **b.** $25^2 = 625$ **c.** $35^2 = 1225$ **d.** $45^2 = 2025$

5. Write the squares of the numbers ending in 5 from 55 through 95.

6. If two factors differ by 2 you can write their product if you know the square of the number between them. For example, $41 \times 39 = 1599$. We know that $40^2 = 1600$; therefore, $41 \times 39 = 40^2 - 1$. Does $51 \times 49 = 50^2 - 1$? Test the rule with a few more examples.

7. Find the following products without written computation.

a. 61×59	**e.** 46×44	**i.** 86×84
b. 81×79	**f.** 76×74	**j.** 24×26
c. 89×91	**g.** 99×101	**k.** 106×104
d. 69×71	**h.** 96×94	**l.** 31×29

A constant and adequate supply of goods for the consumer requires not only a source of raw materials, labor, machinery, and other resources, but also a source of funds to finance the steps in production and distribution. It is the function of the *commercial bank* to supply these funds. The bank also provides many other services for its customers, such as checking and savings accounts, consumer loans, selling bonds and travelers' checks, and so on. Its original and primary service, however, is to create and control *commercial credit*. A typical transaction will illustrate the nature and importance of this service.

Since a bank has an overhead and must make a profit, the bank charges *interest* on the money it lends. The interest is computed as a per cent of the money it lends.

Last September, Mr. Henderson had a chance to buy 400 bushels of sweet potatoes at a very low price if he could pay cash. While he could make a good profit in a short time if he could buy the potatoes, he did not have that much cash. He explained to his banker that he could pay off the loan with interest with what he received from selling the sweet potatoes and still make a profit on the deal.

When he borrowed the money, Mr. Henderson signed a promise to repay the money with interest at a stated time. This is a *promissory note:*

$300.00 St. Paul, Minn. *Sept. 10, 1965*

Thirty days after date *I* promise to pay to the order of

EAST SIDE STATE BANK OF ST. PAUL

Three hundred and no/100 dollars

For value received, with interest at *five* per cent per year.

Royal Fruit and Vegetable Market

By *A. B. Henderson*

1. The person who signs the note is called the *maker*. Who is the maker?
2. The amount borrowed is called the *principal*. What is the principal?
3. The *term* is the length of time for which the money is borrowed. What is the *term* of the note?
4. The *date of maturity* is the date when the note is due and payable. What is the date of maturity of the note?

5. The maker agrees to pay the lender interest for use of the money. The *rate of interest* is the per cent of the principal to be paid as interest each year. If the money is loaned only for a month, then only $\frac{1}{12}$ of the yearly interest is paid. Usually, a year is counted as 360 days to simplify the calculation. Then 60 days are $\frac{1}{6}$ of a year; 90 days are $\frac{1}{4}$ of a year; and the fractions are easy to work with.

EXAMPLE

Find the interest on \$350 for 1 year at $6\frac{1}{2}\%$.
$6\frac{1}{2}\% = 6.5\%$.
The interest on \$350 for 1 year at $6\frac{1}{2}\%$ is

$$0.065 \times \$350 = \$22.75$$

Find the interest on each of the following sums for one year.

a. \$500 at 4%
b. \$3000 at 3%
c. \$475 at $4\frac{1}{2}\%$
d. \$250 at 6%
e. \$865 at 5%
f. \$1355 at $2\frac{1}{2}\%$
g. \$270 at 5%
h. \$14.50 at 4%
i. \$367.80 at $3\frac{1}{2}\%$

6. In the note Mr. Henderson agrees to pay interest at the rate of 5 per cent of the principal each year. How much would that be if the note ran for a full year?

7. The money is borrowed for only 30 days which is $\frac{1}{12}$ of a year. What interest does Mr. Henderson pay?

8. Suppose you borrowed \$100 from the Eureka National Bank as follows:

Date, April 1 (this year)
Term, two months
Rate of interest, 5% per year

Make out the note with yourself as maker.

9. What would the interest amount to, in Exercise 8, counting two months as one-sixth of a year?

10. With yourself as maker, make out a note for the following loan:

Principal, \$500
Date, April 15 (this year)
Term, 90 days
Rate of interest, 4% per year

What is the date of maturity?

11. How much interest will be due on your note when it matures?

12. What is the amount required to pay it off — principal plus interest?

To find the interest for a term of 1 year on a given principal, you multiplied the rate of interest times the principal. Using the language of algebra, we can express this relationship as a formula. We will let i represent the amount of interest, p the principal, and r the rate of interest: The Formula becomes

$$i = p \times r \qquad \text{or} \qquad i = pr$$

If the loan is made for more or less than 1 year, we have to consider also the time as a factor, as in solving the Problems on page 399. If the term of the loan was 6 months, you multiplied $\frac{1}{2}$ (6 months = $\frac{1}{2}$ year) times the product of the principal and rate. If the time was 3 months, you used $\frac{1}{4}$ as a factor. Similarly, if the term was 1 month, $\frac{1}{12}$ was the factor. Therefore, the Formula now becomes

$$i = p \times r \times t \qquad \text{or} \qquad i = prt$$

where t is the time in years or fraction of a year.

EXAMPLES

1. Find the interest on \$600 at 4% for $2\frac{1}{2}$ years. The formula is: $i = prt$

$$p = \$600,\ r = \tfrac{4}{100},\ t = 2\tfrac{1}{2} = \tfrac{5}{2}$$
$$i = \$600 \times \tfrac{4}{100} \times \tfrac{5}{2} = \$60$$

2. Find the interest on \$900 at $4\frac{1}{2}$% for 65 days. The formula is: $i = prt$

$$p = \$900,\ r = \tfrac{9}{200},\ t = \tfrac{65}{360}$$
$$i = \$900 \times \tfrac{9}{200} \times \tfrac{65}{360} = \$7.31$$

Use the formula to find the interest in these problems. The principal, time, and rate are given.

1. \$150, 60 days, 5%
2. \$2500, 3 years, $3\frac{1}{2}$%
3. \$246, 90 days, 6%
4. \$3600, 75 days, $4\frac{1}{2}$%
5. \$640, 80 days, 5%
6. \$480, 2 years, $3\frac{1}{2}$%
7. \$750, 66 days, 4%
8. \$600, 72 days, 6%
9. \$750, 39 days, $5\frac{1}{2}$%
10. \$450, 66 days, 3%
11. \$90, 80 days, $4\frac{1}{2}$%
12. \$3600, 50 days, 6%
13. \$720, 60 days, 6%
14. \$150, 72 days, $4\frac{1}{2}$%
15. \$360, 90 days, $5\frac{1}{2}$%
16. \$660, 78 days, 6%
17. \$750, 60 days, 8%
18. \$1500, 2 years, $4\frac{1}{2}$%
19. \$385, 90 days, 6%
20. \$2400, 75 days, $7\frac{1}{2}$%
21. \$720, 80 days, $5\frac{1}{2}$%
22. \$900, 66 days, 7%

1. Examine this promissory note:

$900.00	April 6, 1965
Ninety days after date I promise to pay to the order of Harvey Jones	$900.00
Nine Hundred and no/100	Dollars
plus interest at Five per cent (5%) per year.	
	Eric Swanson
415 Palm St., Grandview, Missouri	

a. Who is the maker of this note?
b. What is the date of maturity?
c. What is the term of the note?
d. What is the rate of interest?
e. How much is the principal?
f. How much interest will it earn?

2. Make out a note, dated today, for $250 to the Citizens National Bank at 6% interest. Calculate the amount due one year from today.
3. If you paid the interest on the note in Exercise 2 semi-annually (every six months), how much interest would be due at each payment?
4. Suppose you paid the interest on the note in Exercise 2 quarterly (every three months), how much interest would be paid each quarter?
5. Complete the note by making the term 9 months. What amount is needed to pay interest plus principal at maturity?

SPECIAL PROJECT

Send a committee to visit a local commercial bank to find out something about the various services the bank offers to the business man and also to the consumer. The bank will probably have descriptive literature describing the more important services available. Be sure to find out about such things as:

Different kinds of checking accounts
Different kinds of savings accounts
Assistance with investments
Various kinds of loans
Safety deposit vaults

Have the committee bring back business forms that are used in the activities they investigate. After explaining them to the class, have them posted on the bulletin board for investigation.

The Interest Formula

Occasionally we know the interest and wish to find one of the other elements in the Interest Formula — the time, rate, or principal. For this purpose, it is most convenient if we have the Formula in a form where the unknown is alone on one side of the Formula. That is, we must *solve* the Formula for the unknown.

We can do this because the Formula has among its elements the factor-factor-product relationship that we are familiar with. Therefore, the Interest Formula has four equivalent forms which can be derived readily. They provide us with a formula convenient for finding time, rate, and principal in addition to the one for finding interest.

When it is important to find how long it will take a given sum of money to earn a certain amount of interest when loaned at a specific rate, we need the Interest Formula with t alone on one side. That is, we must *solve* the Formula for t. Let us examine the Formula in its factor-factor-product relationships.

EXAMPLE

Solve $i = prt$ for t. $t = ?$

$i = (p \times r) \times t$. We can use the associative property to group $p \times r$ as one factor.

Hence $p = x \times y$

$p \div x = y$

Then $\dfrac{i}{p \times r} = t$

Or, more commonly, $t = \dfrac{i}{p \times r}$

1. Mr. Smith borrowed \$500 at 6%. How long will it take for the interest to amount to \$10? Substitute in the Formula, and solve.

$$t = \frac{i}{p \times r} \qquad i = \$10; \qquad p = \$500; \qquad r = 6\%$$

2. Mr. Anderson borrowed \$800 at 5%. How long will it be before he owes \$30 interest?

3. Mr. Jensen borrowed \$300 at 6% interest. On the date of maturity he repaid the loan with \$27 interest. What was the term of the loan?

4. Sometimes it is important to determine the rate of interest, r, being charged on a loan. This is especially the case with personal loans,

where the borrower is using the money not for business purposes but for household expenses, or for purchasing a car, and so on. Frequently in such transactions there are extra fees, such as clerical charges, that should be added to determine the cost of the loan. To determine the rate of interest we need to solve the Formula for r. Using the method in the Example, solve the Interest Formula for r.

5. Mr. Adams borrowed \$500 for 90 days and paid \$8.75 in interest. What rate of interest did he pay?

We use the formula

$$r = \frac{i}{pt}$$

where $i = \$8.75$; $p = \$500$; $t = \frac{90}{360}$.

6. Mr. Olsen paid \$1.50 interest on a 2-month loan of \$150. What was the rate of interest?

7. Mr. Jefferson borrowed \$200 for 3 months at a personal loan agency. In addition to \$4 interest, he had to pay \$1 for the cost of having his credit rating investigated. To find the total interest paid, the \$1 must be added to the \$4. What was the actual rate of interest?

8. Henry wanted to borrow \$150 for 6 months to purchase a used motorcycle. He found he would have to pay interest at the rate of 8% a year plus a \$4 *service charge.* A service charge is an additional expense for investigating credit rating, preparing papers, and so on. These are to be added to the stated interest to find the total interest. What was the actual rate of interest?

9. When we need to find the principal that, at a given rate for a specified time, will yield a given sum as interest, we use the Formula in a form that is solved for p. Using the method in the Example, solve the Interest Formula for p.

10. What sum must be invested at 4% to yield \$25 interest each 3 months?

Replace each question mark with the correct answer.

	Principal	*Rate*	*Time*	*Interest*
11.	\$ 750	?	90 da.	\$11.25
12.	\$2000	5%	?	\$50
13.	\$ 600	?	60 da.	\$ 4
14.	\$ 400	5%	90 da.	?
15.	\$ 500	3%	?	\$10
16.	?	3%	120 da.	\$ 2
17.	?	4%	180 da.	\$ 6
18.	\$ 100	?	180 da.	\$ 3

This puzzle-type problem with missing digits will give you a chance to practice the steps for solving mathematical problems. Remember that the secret is to find as many clues as you can until you see a promising lead to the solution.

STEPS FOR SOLVING MATHEMATICAL PROBLEMS

1. Understand the problem.
2. Analyze the data.
3. Discover new facts.
4. Follow up and verify promising leads.
5. Review your solution.

In the division problem below on the right, all but five digits were erased. However, from these five a puzzle expert can find all the rest. Can you?

```
      x 4 x
  x)x 8 x x
    x x
      1 4
      x x
        x 1
        x x
```

Clue 1: Notice in the second division that we divided an unknown divisor into 14 and obtained 4 as a quotient. This suggests that the divisor is 3. Why?

Clue 2: If the divisor is 3, then the second partial product (the number named under 14) must be 12.

Clue 3: Now we know that 12 from 14 is 2. Therefore, the last division must be 3 into 21. The last quotient must be 7.

Clue 4: Now let's go back to the first partial product. The right-hand digit must have been 7, since $8 - x = 1$. The smallest multiple of three ending in 7 is 27. Therefore the first quotient must be 9.

Try the following problems. See if you can locate the clues. The problems are very difficult if you try to solve them only by guessing.

```
        4 x 5 1
   2x)8 x x x 2
      x x
        1 x x
        x x x
            x 2
            2 x
```

Clue 1: The last number named in the quotient is 1. Since there is no remainder, the final partial product must be 22. Then the number named in the divisor in the ones place must also be 2.

Clue 2: The first partial product results from multiplying 4 and 22. Thus, the first partial product must be 88 and the second number named in the dividend must be 9.

Clue 3: The second number named in the quotient must be zero. Why? What are the other missing digits?

Clue 1: The product of 8 and xx is a number named by two digits. Then xx must be 10, 11, or 12. Why?
Clue 2: It cannot be 12. Why?
Clue 3: It cannot be 10. Why?

```
      8 x x x
x x)x 6 x x x
    x x
      x x
      x 8
        1 x
        x x
```

1.
```
        9 x 5 x
x x)x x x x x x
    x x
    x x x
      x x
        5 x
        x x
          1 x
          x x
```

2.
```
      7 x 8 x
x x)x 2 x x x
    x x
      1 x x
      x x x
          1 x
          x x
```

3.
```
      4 x 5 x
2 x)x x x x x
    x 6
      3 x
      x x
      x x x
      x x x
          2 x
          x x
```

4.
```
      3 x 8 x
3 x)x 8 x x x
    x x
      x x x
      x x 6
          6 x
          x x
```

5.
```
      4 x x x
x 5)x x x 3 x
    x x
      1 x
      x x
        x 3 x
        x x x
```

6.
```
          6 x x
x 4 x)x x x x x
      x x x
          2 x x
          x x 4
```

7. Given the exercise: x4 ÷ x = x4. Find the digits that will make this a true statement. Explain each of the clues that led you to the discovery of the correct digits.

8. Can you complete a division exercise and then erase several digits to make a problem similar to Exercises 1–6? Try it.

In the price lists below, you will see some advertisements from the newspaper. Round all answers to the nearest per cent.

1. **a.** What is the largest discount (in dollars and cents) on furniture?
   b. What per cent of the regular price is this?
2. **a.** What is the smallest discount on furniture?
   b. What per cent of the regular price is this?
3. **a.** What is the largest discount on rugs?
   b. What per cent of discount is this?
4. **a.** What is the smallest discount on rugs?
   b. What per cent of discount is this?
5. The advertisement on silver specifies a 30% discount. Check a few of the discounts to see if they are about 30%.

Furniture Sale

Article	Was	Now
Drop Leaf Table. 42″x26″	169.95	105.00
40″x60″ Extension Table	139.00	99.00
Slat Back Side Chairs	22.50	14.90
Windsor Side Chairs	26.50	17.50
Windsor Side Chairs	29.95	19.75
Mates Chairs	32.50	21.50
Captains Chairs	42.50	27.95
Governor Carver Arm Chairs	47.95	39.50

Save 15% to 35%

"Naugahyde-leather" FOAM CUSHION LOUNGE CHAIRS. Choice of colors. Reg. 119.50 ..... 59.95

3-PC CURVED SECTIONAL in "naugahyde-leather." Bumper end. 90-degree curve. Reg. 359.00...... 279.95

Extra long DECORATOR SOFAS — loose foam cushions on back and seat. Reg. 319.00................ 229.95

Modern 3 CUSHION FOAM SOFA with new sloping arm. Reg. 239.95. 179.95

Beautiful French provincial or colonial sofa. Shaped back. 1 only. Reg. 299.00.................... 199.00

100″ king size foam sofa. Pillow arm. Choice of colors. Reg. 299.00...... 179.95

Foam rubber provincial chair. 1 only to sell. Reg. 149.95.......... 89.95

Rug Sale

Size	Color	Was	Now
12x10.1	Smoky texture	139.50	59.95
9x19	Green wool wilton	249.50	119.95
12x7.4	Sandalwood tweed	49.50	19.95
12x9.6	Rosewood wool tex	104.50	49.95
15x7.5	Green, white tweed	92.50	39.95
15x16	Brown tweed, wool nylon	219.95	109.50
15x16	Thick plush gold, wool	432.50	214.95
15x14.6	Brown, shag, wool nylon	274.50	129.95

Save 30% on Silver

Place setting pieces	Reg.	Spec.
Teaspoon	4.25	2.98
Place fork	6.50	4.55
Place knife	6.25	4.38
Salad fork	5.50	3.85
Cream soup	5.50	3.85
Dessert spoon	5.50	3.85
Butter spreader, hh	4.75	3.33

Part One

A. Express each of the following as a per cent rounded to the nearest tenth of 1% where necessary.

1. 0.23	**3.** 0.049	**5.** 0.456	**7.** $\frac{5}{8}$	**9.** $\frac{5}{16}$
2. 0.009	**4.** 2.4	**6.** $\frac{3}{5}$	**8.** $\frac{17}{12}$	**10.** $\frac{16}{25}$

B. Find the value of N in each of the following:

11. 48 is $N\%$ of 80

12. $N\%$ of 16 is 12

13. N is 25% of 180

14. 20% of N is 40

15. $\frac{1}{2}\%$ of 800 is N

16. 3.5% of 60 is N

C. Calculate the interest in each of the following:

	Prin.	*Rate*	*Time*
17.	$800	6%	72 days
18.	$750	4%	90 days
19.	$ 450	5%	90 days
20.	$1200	3%	60 days

D. Do You Understand the Relationships in Selling?

Some of the statements below are true, and some are false. List the numerals 1 through 12 on a sheet of paper. Carefully read each statement. If it is true, write T on your paper after the corresponding numeral. If it is false, write the correct statement after the corresponding numeral on your paper.

1. To find the margin, subtract the cost from the selling price.

2. If the overhead is greater than the margin, the merchant has a profit.

3. You can find the profit by adding the overhead to the cost, and subtracting this sum from the selling price.

4. The margin can be greater than the cost.

5. The margin can be greater than the selling price.

6. If the overhead is less than the margin, the merchant has a loss.

7. You can find the selling price by adding cost, overhead, and profit.

8. The margin is equal to profit plus overhead.

9. The margin can be less than the markup.

10. $S = C + M$

11. $C = S - M$

12. $C = S + O + P$

13. $M = C - S$

14. $L = C - S$

Part Two

A. Each of the following questions is correctly answered by one of the words or phrases that follow it. Write the numerals 1 to 10 on a sheet of paper. After each numeral write the letter to indicate which word or phrase correctly answers the question.

1. What do you call the per cent of sales that a salesman receives as his earnings?
a. margin **b.** commission **c.** profit **d.** wages

2. What is the balance of the margin after the overhead has been paid?
a. gross sales **b.** overhead **c.** profit **d.** cost

3. What do you call the difference between selling price and cost?
a. margin **b.** overhead **c.** profit **d.** discount

4. What is the reduction from the regular price of an article on sale?
a. commission **b.** loss **c.** cost **d.** discount

5. What term is used when the cost plus overhead exceeds the selling price?
a. loss **b.** discount **c.** gross sales **d.** depreciation

6. What is the price the customer pays for an article?
a. gross sales **b.** selling price **c.** sales slip **d.** profit

7. What do you call the total money taken in by a merchant over a period of time?
a. gross sales **b.** overhead **c.** profit **d.** commission

8. Which of these is equal to cost + overhead + profit?
a. margin **b.** selling price **c.** operating expenses **d.** discount

9. Which of these is equal to profit + overhead?
a. margin **b.** gross sales **c.** commission **d.** loss

10. Which of these items is *not* included in overhead expenses?
a. heat **b.** wages **c.** rent **d.** cost of merchandise

B. Graphs

1. In a recent report, the confectionery store sales dollar was divided as follows: cost of merchandise, 60¢; overhead, 31¢; profit, 9¢. Show these facts on a divided bar graph.

2. In the same report, the grocery store sales dollar was divided as follows: cost of merchandise, 75¢; overhead, 18¢; profit, 7¢. Show these facts on a circle graph.

3. Using the data listed in Exercise 1, find the margin if the confectionery store has an income of $625 on a particular Saturday.

4. Which store has the greater per cent of margin?

Part Three

1. A supermarket purchased 1500 bushels of potatoes for \$3000. The potatoes were sold at \$2.50 per bushel. What per cent of the selling price was the margin?
2. Apples that cost \$2.50 per 100 lb. were sold at 3¢ per lb. What per cent of the cost was the margin?
3. Grapefruit were sold at 6¢ per lb. The margin was 20% of the selling price. How much per lb. did they cost?
4. The Adams Realty Company receives a 7% commission based on the rents which they collect. Last month, the rents they collected in one apartment building totaled \$3000. How much was the commission?
5. A salesman working on a commission of 15% had sales last week of \$975. How much was his commission?
6. Last week, Mabel earned a commission of \$6 selling subscriptions to a magazine. She sold \$40 worth of subscriptions. What was the rate of commission?
7. Last month Mr. Hall had gross sales of \$4250. The cost of merchandise sold was \$3400. His expenses were \$595. What per cent of the sales were his profits?
8. A supermarket purchases eggs at 30¢ a dozen. The selling price is set to provide a margin of 25% of the selling price. How much do the eggs sell for?
9. Oranges are purchased for \$6 per 100 lb. They sell at a margin of 40% of the selling price. What is the selling price?
10. Jim receives a commission of 10% for selling newspapers. Last week, he earned \$12. What did his sales amount to?
11. A real estate company that charges 6% commission on sales received \$180 on a sale of property. What did the property sell for?
12. The Campus Clothing Store had a sale at which all merchandise was reduced 15% in price. Jerry bought a sport coat for \$34. What was the regular price of the coat?
13. The Book Shop had a clearance sale at which prices on all merchandise were reduced 20%. Mabel bought a history book for \$2.40. What was the regular price of the book?
14. What did a loose-leaf notebook sell for at the sale if its regular price was \$1.80?
15. Mike saved 45¢ by purchasing a fountain pen at the Book Shop during the sale. What did he pay for the pen?

CHAPTER ELEVEN

PROBABILITY

WORDS TO WATCH FOR

circular permutations
combinations
dependent events
factorial
mutually exclusive
permutations
probability
sample space

The *theory of probability* is one of the most fascinating and entertaining branches of mathematics. It is used to compute the chance, or *probability,* that a given event will or will not occur. It serves not only to predict events but to discover new facts. For this reason, it is used in communications, marketing, nuclear physics, insurance, and in many other phases of science and everyday life.

The theory of probability had a disreputable beginning in gambling. However, games of chance still provide many of the simplest and most clearcut applications of probability. Many of the common occurrences of life have elements of chance that need to be taken into account. People in certain occupations, for example, have to pay more money for life insurance because of the risks involved in their jobs. Name some occupations for which, in your opinion, the cost of life insurance would be high.

Before the start of a football game the referee usually calls the captains of the two teams onto the field to decide two questions:

(a) Which team will kick and which will receive?
(b) Which goal will each team defend?

To settle this problem impartially the referee tosses a coin, asking the captain of the visiting team to call "heads" or "tails" as his

prediction of the side that will lie upward when the coin has fallen. If he predicts correctly he may make his choice on either of the two questions as he chooses.

The probability of the captain winning the toss is the number of times his choice could occur as compared to the number of ways the coin can fall. If it is known that an event is equally likely to occur in several different ways, then the probability of a specified event occurring in any trial is expressed as a ratio:

$$p = \frac{s}{t}$$

where s = the number of ways in which the specified event can occur
and t = the total number of ways the equally likely events can occur

No one doubts that tossing a "fair" coin gives an equal chance to both teams for a choice of goals or of kicking or receiving. A "fair" coin is one that is well balanced and unlikely to stand on its edge. Since it can fall only in two ways, heads or tails, and each way is equally likely, then $t = 2$ and $s = 1$. Therefore, the probability of either event is $\frac{1}{2}$. We might express this another way by saying that there is a "50-50" chance of heads or tails on each throw. Suppose we have a bag containing a red marble, a blue one, and a white one. We are to pick out one marble, so $s = 1$. Here any one of the three results is equally likely, so $t = 3$. Therefore the probability of picking any one color (such as red) is the ratio:

$$p = \frac{\text{the number of ways of picking a red marble}}{\text{the number of ways a marble can be picked}} = \frac{s}{t}$$

Then $$p = \tfrac{1}{3}$$

A *die* is a cube with a different number of dots, from 1 to 6, on each face. If we roll a fair die, what is the probability that 4 will turn up? (Hereafter, we will not use the word fair because we will only be concerned with coins, dice, etc., that are fair.)

Since any one face is equally likely to turn up on any toss, and there are 6 faces on the die, then $t = 6$. Only one face of the die is 4, so $s = 1$. Therefore, the probability of rolling a 4 in any one roll is $\frac{1}{6}$.

1. What chances, risks, or probability are involved in each of the following events?

   a. Betting on a horse race
   b. Buying stock for an investment
   c. Buying a used car

   Can the risk be lessened by increased knowledge about what you are doing?

2. Use three coins for this experiment. Toss all three coins into the air twenty times. On a chart like the one below, record the number of heads or tails that turn up. Which manner of falling occurred the most? the least?

Manner	*How Many Times*
3 heads	?
2 heads, 1 tail	?
1 head, 2 tails	?
3 tails	?
Number of tosses	?

3. What is the probability of each manner of falling? How do the results of this experiment agree with the computed probability?
4. If two dice are tossed, how many distinct pairs of numbers are possible?
5. If two dice are tossed, in how many ways can the total on the two dice equal 9?
6. In how many ways can the total on both dice equal 12?
7. In how many ways can the total on both dice equal 7?
8. What is the probability of rolling a total of 12 with two dice? a total of 9? a total of 7?
9. There are three black marbles and two white marbles in a bag, and one marble is drawn at random. What is t? What is s for the black marble? for the white marble?
10. What is the probability of drawing a white marble in Problem 9? the probability of drawing a black marble? Do you agree that the probability of drawing a black marble should be greater?
11. If one card is drawn from a set of cards on which are written the letters of the English alphabet, what is the probability of drawing a vowel (a, e, i, o, or u)? of drawing a consonant?
12. If one card is drawn from the set in Exercise 11, what is the probability that it has one of the letters of the word *sequoia* written on it? What is the probability that it has not?
13. If one card is drawn from a set of cards on which are written the numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, what is the probability that it is one of the digits of the numeral 340,671? What is the probability that it is not?
14. In terms of probability, an event which is certain has a probability of 1, and an impossible event has a probability of 0. How is this illustrated in Problems 11, 12, and 13?

How Many Ways?

Suppose a monkey is taught to place three cards, each having one of three letters A, T, or C, written on it, in a row. What are the chances that he will spell CAT in any one trial?

First let us list all the different ways in which the three cards can be arranged. They are: {ACT, ATC, CAT, CTA, TAC, TCA}. There are six different ways the monkey can arrange the cards, and each is equally likely. The set of possible ways in which an event can happen in any experiment is called the *sample space* of the experiment.

1. The experiment is to determine the probability of the monkey spelling CAT in one trial. How many elements of the sample space spell CAT?

2. If we had the monkey repeat the experiment several times, in what fraction of the total trials would he spell CAT, on the average?

3. If the monkey repeated the experiment 84 times, what would you predict as the number of times he would spell CAT?

4. Suppose the three letters written on the cards were O, W, and T. What is the sample space for arranging the three letters in a row?

5. How many common words are in the sample space of the experiment in Problem 4?

6. If the monkey arranged the letters O, W, and T in a row 33 times, how many times do you predict he would spell a common word? How many times do you predict he will not spell a common word?

7. Try one of the above experiments, or a similar one, by blindfolding your partner and letting him act as the one who performs the experiment.

8. Can you verify your predicted results better with just a few trials or with many?

9. Suppose you conduct an experiment by throwing a die. If we consider only the number of dots that roll face up, list the set of numbers which is the sample space for the experiment.

10. In what fraction of the tosses, on the average, will an even number turn up? an odd number? a number divisible by 3?

11. If you toss two dice (such as a and b) how many numbers can turn up on b with any number turned up on a?

12. What is the total number of ways in which the two dice can fall?

13. If you add the number of dots turned up on the two dice, in how many different ways can they fall so the total is 7?

As we have seen, it is convenient to express the probability that an event will happen in a certain way by using a ratio. Let us use a ratio again to define the probability that a certain event will occur in an experiment.

Let p represent the probability a certain event will occur in an experiment.

Let s represent the number of ways a certain event can occur successfully in an experiment.

Let t represent the sample space of the experiment (all the possible occurrences in the experiment).

The probability of the occurrence of a successful event is the ratio of the number of ways the event can occur successfully to the number of possible ways the event can occur.

$$p = \frac{s}{t}$$

EXAMPLE

Suppose you toss two coins. What is the probability that both will come up heads? Let us call the two faces of the first coin H_1 (heads) and T_1 (tails). Call the faces of the second coin H_2 and T_2. The sample space for this experiment is $\{T_1T_2,\ T_1H_2,\ H_1T_2,\ H_1H_2\}$. Therefore, $t = 4$. In the sample space there is only one element that represents a successful event (H_1H_2). Therefore, $s = 1$. Then, $p = \frac{s}{t} = \frac{1}{4}$.

Use the Formula above to do the following problems.

1. Using the sample space in the above Example, what is the probability that both coins will come up tails?
2. What is the probability that one will be heads and one will be tails?
3. Three coins are tossed. Using H_1, T_1, H_2, T_2, H_3, and T_3 to represent the faces of the three coins, write all the elements in the sample space of this experiment.
4. How many of the elements consist of two heads and one tail?
5. What is the probability of tossing two heads and one tail?
6. How many of the elements in the sample space consist of two tails and one head?
7. What is the probability of tossing two tails and one head?
8. What is the probability of tossing three heads? three tails?

9. The Table below shows the 36 possible sums in the sample space when two dice are thrown. What is the probability of throwing a number greater than 7 in tossing two dice?

Number up on Die #1

Number up on Die #2	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

10. What will be the value of t in expressing the probability of tossing any number with two dice?

11. Explain how you can use the Table to find the value of s in expressing the probability of tossing any number with two dice.

12. What is the probability of tossing each of these numbers with two dice: 5? 2? 8? 12?

13. What is the probability that, if two dice are tossed, the number turning up will be greater than 1 and less than 13?

14. What is the probability that, if two dice are tossed, the number turning up will be greater than 12?

15. If the probability of winning a game is $\frac{1}{3}$, what is the probability of losing the game?

16. A bag contains 3 white balls and 2 black balls. In how many ways can you draw from the bag:

a. 1 white ball?
b. 1 black ball?
c. 2 white balls?
d. 3 white balls?
e. 1 white ball and 1 black ball?
f. 2 black balls?

17. In how many ways can you draw two balls from the bag?

18. What is the probability that if two balls are drawn from the bag in Exercise 16 both will be black? that both will be white?

19. Mary had a red dress, a blue dress, and a green dress. Helen had a red dress, a yellow dress, and a brown dress. If the wearing of any of the dresses is equally likely for both girls, what is the probability that on any given day, they will both wear red dresses?

20. Bill and Fred were each given a choice of the same four books from which to make a book report. What is the probability that they will both report on a given book?

In many situations it would be tedious and difficult to list all of the elements in a sample space. Let us examine some other methods of finding how many ways a specified event can occur. This is important not only in the use of the rules of probability but for its own sake as well.

Consider this problem. Mary has 3 pairs of shoes, 4 dresses, and 5 sweaters. With each pair of shoes she can choose any one of 4 dresses. Therefore she can have 3×4 or 12 different arrangements of shoes and dresses.

If we let S_1, S_2, and S_3 represent the 3 different pairs of shoes, and D_1, D_2, D_3, and D_4 represent the 4 different dresses, the twelve different arrangements of shoes and dresses would be:

S_1D_1	S_2D_1	S_3D_1
S_1D_2	S_2D_2	S_3D_2
S_1D_3	S_2D_3	S_3D_3
S_1D_4	S_2D_4	S_3D_4

With each arrangement of shoes and a dress she can select any one of 5 sweaters. Therefore she will have a total of $(3 \times 4) \times 5$ or 60 different arrangements of the 3 items.

This leads us to an important Principle of this chapter.

Principle: Suppose one event can occur in A different ways, a second event can occur in B different ways after the first event occurs, and a third event can occur in C different ways after the first two occur, etc. Then the number of different ways all the events can occur is $A \times B \times C$, etc.

EXAMPLES

1. There are 4 different routes from Town X to Town Y. There are 7 different routes from Town Y to Town Z. In how many different routes can a person travel in going from Town X to Town Y to Town Z?

Solution: (a) For each of the 4 routes from Town X to Town Y, there are 7 different routes to Town Z. Therefore, there are $4 \times 7 = 28$ different routes from Town X to Town Y to Town Z.

(b) Apply the Principle above:

$$N = 4 \times 7$$
$$N = 28$$

Answer: 28 different routes

2. How many different four digit numerals can be made using only the digits 1, 3, 5, 6, 8, and 9, if each digit of the four digit numeral is to be different?

Solution: Consider the choices for the units digit of the four digit numeral. It can be any one of the 6 (__ __ __ $\underline{6}$). Then since one of the six choices has been picked for the units digit, the tens digit can be any one of the 5 remaining. In like manner the hundreds digit can be any one of the 4, and the thousands digit can be any one of the 3. The number of choices is ($\underline{3}$ $\underline{4}$ $\underline{5}$ $\underline{6}$). The total $= 3 \times 4 \times 5 \times 6 = 360$ different four digit numerals.

3. How many different four digit numerals can be obtained from Example 2 if the digits can be repeated?

Solution: Each of the digits of the four digit numeral can be repeated and can be any one of the 6 choices.
By the Principle above, the total $= 6 \times 6 \times 6 \times 6 = 1296$.

Use the Principle on page 416 to solve the following problems.

1. In how many different ways can 5 boys and 2 girls be seated in a row of 7 seats? Would the results be different if the 7 persons were all boys or all girls?
2. How many different seating arrangements can be obtained in Problem 1 if the first and last seats are to be filled by a boy? if each is to be filled by a girl?
   HINT: Fill in the first and last seats first.
3. From the numerals 1, 2, 3, 4, and 5, how many three digit numerals can be formed, if all 3 digits are to be different?
4. How many *odd* numbers can be named by three digit numerals if all 3 digits (Problem 3) are to be different? how many even?
5. A certain lunch consists of a sandwich, a beverage, and a dessert. If there is a choice of 4 different kinds of sandwiches, 3 different beverages, and 3 different kinds of desserts, how many different lunches can be served?
6. In Problem 5, if each lunch is equally desirable, what is the probability that a lunch consisting of a tuna sandwich, milk, and an apple be chosen at any one time?
7. Harry, Bill, Fred, and John are infielders on the baseball team. In how many different ways can the baseball coach fill the four infield positions from these 4 boys?

Permutations

The different ways of arranging the elements of a set, in order from first to last, are called the *permutations* of the set. Here are the permutations of the three letters r, s, and t taken three at a time:

$r\ s\ t$	$s\ t\ r$
$r\ t\ s$	$t\ r\ s$
$s\ r\ t$	$t\ s\ r$

If we take four different letters r, s, t, and u and list all the permutations of them taken four at a time, we will obtain 24 different arrangements. They are:

$r\ s\ t\ u$	$s\ r\ t\ u$	$t\ r\ s\ u$	$u\ r\ s\ t$
$r\ s\ u\ t$	$s\ r\ u\ t$	$t\ r\ u\ s$	$u\ r\ t\ s$
$r\ t\ u\ s$	$s\ t\ r\ u$	$t\ s\ r\ u$	$u\ t\ r\ s$
$r\ t\ s\ u$	$s\ t\ u\ r$	$t\ s\ u\ r$	$u\ t\ s\ r$
$r\ u\ s\ t$	$s\ u\ t\ r$	$t\ u\ r\ s$	$u\ s\ t\ r$
$r\ u\ t\ s$	$s\ u\ r\ t$	$t\ u\ s\ r$	$u\ s\ r\ t$

The number of different permutations can be found easily by using the Principle on page 416. Solving for the number of permutations of the 4 different letters taken 4 at a time, we can say that the first letter can be any one of 4, the second letter can then be any one of the three remaining letters, the third letter can be any one of the remaining 2, and the fourth letter must be the only one remaining. The total is $4 \times 3 \times 2 \times 1$ or 24 different permutations.

How about the set of all the permutations of these same four letters taken 3 at a time? The Principle on page 416 gives us a total number of $4 \times 3 \times 2$ or 24. Can you list all 24 permutations?

How about the set of all the permutations of these same four letters taken 2 at a time? Can you list all of them?

In order to set up some formulas for solving permutations concisely, we shall introduce a new mathematical symbol.

If n represents any natural number, then $n!$ (read "n factorial") means

$$n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times (1)$$

or

$$n(n-1)(n-2)(n-3) \ldots (1)$$

The three dots, . . ., above mean that the multiplication continues with $(n-4)$, $(n-5)$, etc., down to the last factor, 1.

EXAMPLES

1. If $n = 4$, then $4! = 4(4-1)(4-2)(4-3)$
$$= 4 \times 3 \times 2 \times 1 = 24$$

Note: The last subtraction is $(4 - 3)$ and not $(4 - 4)$. The last subtraction, $(4 - 3)$, gives us 1, and the definition of $n!$ states that the last factor should be 1. The last subtraction could not be $(4 - 4)$. Why not?

2. If $n = 8$, then $8! = 8(8-1)(8-2)(8-3)(8-4)(8-5)(8-6)(8-7)$
$$= 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40{,}320$$

3. If $n = 8$ and $m = 6$, then

$$\frac{n!}{m!} = \frac{8(8-1)(8-2)(8-3)(8-4)(8-5)(8-6)(8-7)}{6(6-1)(6-2)(6-3)(6-4)(6-5)}$$
$$= \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

Since the numerator and denominator have common factors of 6, 5, 4, 3, 2, 1, we can divide the numerator and denominator by these common factors.

Then
$$\frac{n!}{m!} = \frac{8 \times 7 \times \overset{1}{\cancel{6}} \times \overset{1}{\cancel{5}} \times \overset{1}{\cancel{4}} \times \overset{1}{\cancel{3}} \times \overset{1}{\cancel{2}} \times 1}{\underset{1}{\cancel{6}} \times \underset{1}{\cancel{5}} \times \underset{1}{\cancel{4}} \times \underset{1}{\cancel{3}} \times \underset{1}{\cancel{2}} \times 1}$$
$$= 8 \times 7$$
$$= 56$$

Evaluate the following:

1. $6!$

2. $5! - 6!$

3. $(4!)(2!)$

4. $6! \div 4!$

5. $6! + 3! + 4!$

6. $(5!)(6!)$

7. $9! + 6!$

8. $(3! + 4!)5$

9. $6! \div 7!$

10. $(4!)(6!) + 3!$

11. $7! \div 2!$

12. $9! \div 6!$

In order to represent the number of permutations of n different objects taken r at a time, we shall use the symbol ${}_nP_r$.

EXAMPLES

1. The symbol ${}_4P_3$ means the number of permutations of 4 different objects taken 3 at a time: ${}_4P_3 = 4 \times 3 \times 2 = 24$

2. The symbol ${}_5P_5$ means the number of permutations of 5 different objects taken 5 at a time: ${}_5P_5 = 5 \times 4 \times 3 \times 2 \times 1 = 120$

3. The number of permutations of 5 different objects taken 3 at a time may be abbreviated as ${}_5P_3$. ${}_5P_3 = 5 \times 4 \times 3 = 60$

Do you see the pattern in the Examples on page 419? As shown in the Examples, the Formula for ${}_nP_r$ is:

$${}_nP_r = n(n-1)(n-2)(n-3)\ldots\big((n-r)+1\big)$$

We can demonstrate the correctness of this Formula with our Principle.

EXAMPLE

How many different permutations can we have from 8 different letters taken 5 at a time?

(a) *Solution by our Principle:* The first letter can be any one of 8 letters; the second can be any one of 7; the third can be any one of 6; the fourth can be any one of 5, and the fifth can be any one of 4. Therefore, the total will be:

$$8 \times 7 \times 6 \times 5 \times 4 = 6720$$

(b) *Solution by the Formula:*

$$\begin{aligned} {}_nP_r = {}_8P_5 &= 8(8-1)(8-2)(8-3)\big((8-5)+1\big) \\ &= 8 \times 7 \times 6 \times 5 \times 4 \\ &= 6720 \end{aligned}$$

Note: Let us examine the term $\big((8-5)+1\big)$ in the above Example. Remember! Always perform the operation within the parentheses first. Since this expression has 2 pairs of parentheses, perform the operation within the parentheses that are inside. That is, first subtract, $(8-5)$. This equals 3. Then add, $(3+1)$. This equals 4.

13. Use the Formula to evaluate the following:

a. ${}_7P_3$ **c.** ${}_8P_3$ **e.** ${}_7P_7$
b. ${}_4P_4$ **d.** ${}_6P_4$ **f.** ${}_{26}P_3$

14. In how many ways can 5 seats be filled by choosing from 8 different people?

15. How many different three letter permutations can be obtained from the entire alphabet if all 3 letters are to be different?

16. How many four digit numerals with all the digits different can be obtained from 1, 2, 4, 7, 8, and 9?

17. How many different three letter arrangements can be obtained from the entire alphabet if the letters can be repeated?
HINT: Use the Principle on page 416.

18. In how many different ways can 10 different books be arranged on a shelf?

Calculation of the number of permutations is often a quick way of determining the total number of elements in the sample space of an occurrence. Where applicable, you can use these methods to compute t in the Formula $p = \frac{s}{t}$, and compute the probabilities. Try it in the following problems.

1. Using just the three letters a, b, and c, what is the probability that an untrained monkey will arrange the letters into the word "cab" in the first try? (First find s and t.)

2. Suppose that the digit 1 is written on a piece of paper, 2 written on another piece of paper, 3 written on another, and 4 written on still another. These 4 pieces of paper are placed in a bowl. Calculate the probability that a blindfolded person will select the pieces of paper in the order 1, 2, 3, and 4.

3. Various signals can be sent in the navy by using flags of different colors arranged in different orders on a line. If each signal is to be four different colored flags, how many different signals can be sent from a set of 7 flags each of a different color?

4. If one of the colors of the signal flags in Problem 3 is yellow, and a signal is chosen at random, what is the probability that the first flag is yellow?

5. Mr. Oliver has five nephews for whom he selected five different presents. In how many different ways might he distribute them?

6. If one of the gifts in Problem 5 is a book, what is the probability that John, one of his nephews, *will not* get the book?

7. Coach Jones has nine players from which to choose a baseball team line up. If the positions are filled at random, how many different line ups can Coach Jones pick? If Harry is one of the players, what is the probability that he will be picked for the outfield?

8. Mary has seven school books in her locker. What is the probability she will first pick her mathematics text and then her science book if she does not look inside first?

9. In how many ways can seven different presents be distributed among seven different children?

10. If Helen likes five of the seven presents and does not like the other two, what is the probability that she will receive a present she likes if they are distributed at random?

1. How many five letter license plates can be made from the alphabet if no letter is repeated?
2. James Brown purchased one of the license plates in Problem 1, selecting it at random. What is the probability that his license spelled his first name?
3. With a set of 5 different signal flags, how many different signals can be made using 3 flags at a time?
4. How many more signals can be made using 5 different signal flags 5 at a time than using 5 different signal flags 2 at a time?
5. Coach Jones has 14 players from which to choose an outfield for his baseball team. From these 14 players, in how many different ways can he fill the 3 positions?
6. A professor has a specially prized set of 7 volumes on a shelf built especially to display them. His maid, who cannot read English, takes them down to dust. She replaces them at random. What is the probability that they are replaced in their original order?
7. Eight golfers are entered in a tournament. In how many different orders can they finish if there is no tie?
8. Ten boys, standing at random in a row, had a picture taken. What is the probability that Fred, one of the boys, was standing third from the left?
9. How many signals can be made from 5 different signal flags if any number from 1 to 5 can be used at one time?
10. A monkey is trained to type at random, the 26 letters of the alphabet on a typewriter. What is the probability that the monkey will type the word "math" in the first four letters?
11. This problem is found in a text published in 1800: Seven gentlemen met at an inn and were so well pleased with their host, that they agreed to tarry so long as they, together with their host, could sit every day in a different manner along a bench at the dinner table. How long must they stay to fulfill their agreement?
12. Fred has 12 pairs of socks in his drawer. Three pairs are brown; four pairs are black; and five pairs are green. He chose a pair in the dark. What is the probability the socks he chose were black?
13. In drawing a card at random from a deck of 52 playing cards, what is the probability of getting either an ace, king, or queen?
14. Fred had to match four dates in a history examination to four different events. He did not know any of the answers. What is the probability that he will guess all four correct answers?

The order of an arrangement is often not important. As an example, a committee consisting of Mary, Bill, and Fred is the same as a committee consisting of Fred, Bill, and Mary. The order of naming the committee does not change the committee.

As you know, if we consider the order in which they are chosen, there are six different permutations of Mary (M), Bill (B), and Fred (F). They are {MBF, MFB, BMF, BFM, FMB, FBM}. Yet if we consider the three people as a group, regardless of order, each of the six permutations is the same group or *combination*. We see that there are $\frac{1}{6}$ as many combinations as there are permutations of the three people.

Now let us consider four objects, such as the four letters A, B, C, and D. There are 24 permutations of these 4 letters taken 3 at a time.

ABC	ABD	ACD	BCD
ACB	ADB	ADC	BDC
BAC	BAD	CAD	CBD
BCA	BDA	CDA	CDB
CAB	DAB	DAC	DBC
CBA	DBA	DCA	DCB

However, if we disregard the order, each of the four columns of different permutations is the same combination of three letters. While there are 24 different permutations, there are only 4 different combinations, ABC, ABD, ACD, and BCD. As you solve the following problems, see if you can find a method for finding the number of combinations that can be made from several different objects.

1. How many different permutations are there for the four letters E, F, G, and H, taken 3 at a time?

2. Consider one of the permutations counted in Problem 1, namely, EFG. List the different ways that E, F, and G can be arranged.

3. For each combination of three letters counted in Problem 1, there are how many different permutations?

4. There are how many times as many permutations of four objects taken three at a time as there are combinations of four objects taken three at a time?

5. Using the same method of analysis, there are how many times as many permutations of five objects taken four at a time as there are combinations of five objects taken four at a time?

6. Using a short cut method, hinted at in Problems 1 through 5, find the number of combinations there are of five different objects taken three at a time.

The Formula for Combinations

The Camera Club consists of 16 members. A committee of 3 members is to be set up, to include a chairman, vice-chairman, and secretary. In how many ways can the committee be set up? Note that here the committees are each an *ordered* set since any 3 members can be rearranged in 3 different offices each comprising a different committee. The number of different committees is the number of permutations of 16 persons taken 3 at a time. We have learned that ${}_{16}P_3 = 16 \times 15 \times 14 = 3360$ different committees.

Now suppose there are no specified offices in the 3-person committee. Then a committee of Smith, Brown, and Jones is the same as Jones, Brown, and Smith. There are no permutations in any of the groups of 3 members. The permutation formula allows for all possible different arrangements in any one group. Since for every group of 3, there are 6 different orders or arrangements, the number of unordered committees would be $\frac{1}{6}$ of 3360 or 560 different groups.

If there are r objects in a group, the number of different permutations of these objects in that group of r objects would be represented by ${}_rP_r$. In this Example:

$$\begin{aligned} {}_rP_r &= {}_3P_3 \\ &= 3 \times 2 \times 1 \\ &= 6 \end{aligned}$$

Therefore, to find the number of unordered combinations, we divide ${}_nP_r$ by ${}_rP_r$. Can you explain why?

A group in which order of arrangement is of no consequence is called a *combination*. The symbol ${}_nC_r$ means the number of combinations of n different objects taken r at a time. Thus, ${}_5C_3$ means the number of combinations of 5 different objects taken 3 at a time.

EXAMPLE

In how many ways can 3 persons be chosen from a group of 9 different persons? This is the number of combinations of 9 different objects taken 3 at a time; that is, ${}_9C_3$.

The computation is ${}_9C_3 = \dfrac{9 \times 8 \times 7}{3 \times 2 \times 1} = 84.$

The Formula in general is

$${}_nC_r = \frac{{}_nP_r}{r!} = \frac{n\,(n-1)(n-2)\cdots((n-r)+1)}{r!}$$

1. Evaluate:

   a. ${}_5C_4$ b. ${}_8C_2$ c. ${}_7C_3$ d. ${}_8C_6$ e. ${}_7C_4$

2. A committee of 3 is to be chosen from a club of 20 members. In how many different ways can the committee be selected?

3. Three names are to be drawn for identical door prizes at a bridge party. Forty names have been entered. How many possibilities for different groups of winners are there?

4. Suppose, in Problem 3, it was specified that there would be a first, second, and third prize. How many different winning combinations would there be?

5. How many different five card hands can be dealt from a deck of 52 playing cards? (The order of dealing is immaterial.)

6. How many different five card hands can be dealt from a deck of 52 playing cards if each is to contain the ace of spades?
   HINT: The answer is the number of four card hands that can be dealt from a deck of 51 cards. Why?

7. How many different five card hands containing both the ace and king of spades can be dealt? (See Problem 6.)

8. How many different five card hands will contain all 4 aces?

9. In how many ways can a group of six people be divided into groups of 4 people and 2 people?

10. In how many ways can two people be selected from a group of six?

11. In how many ways can a group of 4 people be selected from a group of 6 people?

12. Examine your answers to Problems 9, 10, and 11, and explain the relationships you discover.

13. How many three letter combinations can be formed from the English alphabet if:

    a. All letters in each combination are to be the same?
    b. No letter is to be repeated in each combination?
    c. Letters can be repeated in each combination?
    d. The first and last letters are to be the same; the middle letter must be different?

14. How many different sums of money can you make from:

    a. A nickel, dime, and quarter?
    b. A nickel, dime, quarter, and half-dollar?
    c. A penny, nickel, dime, and quarter?
    d. A penny, nickel, dime, quarter, and half-dollar?

15. In how many ways can a committee of 3 persons be made up from a list of 10 persons?

16. In how many ways can a committee be formed, in Problem 15, if 2 people on the list will serve only if together?

17. In how many different ways can the committee be selected if there are two members of the list of 10 who refuse to serve together? HINT: Subtract the number on which they serve together from the total number of possible committees.

18. What is the probability that a hand of two cards dealt from a deck of 52 cards will contain:

    a. 2 spades?
    b. No hearts?
    c. Exactly 1 heart?
    d. The ace of hearts?

19. What is the probability that a hand of four cards will contain one card of each suit?

20. What is the probability that a hand of four cards will contain cards all of one suit?

21. A committee of two Frenchmen (F), two Englishmen (E), and two Germans (G) is to be selected from a group consisting of 10 Frenchmen, 10 Englishmen, and 10 Germans. How many such committees are possible? Replace the question marks to show the choices available for each nationality.

${}_{10}C_2$	?	?

Is ${}_{10}C_2$ the correct symbol? Why? How many choices are available in the second place for each in the first place?

22. In how many ways can 2 jackets, 2 pairs of shoes, and 3 trousers be selected from 3 jackets, 4 pairs of shoes, and 6 trousers?

23. In how many ways can 8 different presents be distributed among 4 children so that each child gets two presents?

24. From a collection of 7 different books, in how many ways can you select: exactly 4 books? 4 or more books? from 1 to 4 books?

25. In how many ways can 9 different books be divided among Jim, Mike, and Charlie so that Jim receives 4 books, Mike, 3 books, and Charlie, 2 books?

26. In how many ways can five guests be chosen from a list of 9 persons for a dinner party?

Independent Acts

Suppose we consider several events A, B, C, etc. The acts are said to be *independent* if the performance of one does not affect the outcomes of any of the others. As an example, let us consider the act of drawing a heart and a club in succession from a deck of cards. If we do not replace the first card we draw, the probability of drawing a particular suit on the second try is changed because there are only 51 cards remaining. These acts are therefore not independent.

For the two successive acts to be considered independent, the first card would have to be replaced before the second card is drawn. Successive acts that are not independent are said to be *dependent.* Here is an important Rule of probability.

Rule 1: Let $p(A)$ be the probability that act A is performed, $p(B)$ be the probability that act B is performed, $p(C)$ be the probability that act C is performed, etc., with A, B, C, etc., each independent acts. Then the probability that A, B, C, etc., will occur in succession is the product of their individual probabilities.

$$p(A, B, C, \text{etc.}) = p(A) \times p(B) \times p(C) \times \text{etc.}$$

EXAMPLES

1. What is the probability of drawing a diamond, d, from one deck of cards and a king, k, from another deck?

Solution: There are 13 diamonds in a deck of 52 playing cards. Therefore:

$$p(\text{diamonds}) = \tfrac{13}{52} = \tfrac{1}{4}$$

There are 4 kings in a deck of 52 playing cards. Therefore,

$$p(\text{kings}) = \tfrac{4}{52} = \tfrac{1}{13}$$

Then

$$p(k, d) = \tfrac{1}{13} \times \tfrac{1}{4}$$
$$= \tfrac{1}{52}$$

2. What is the probability of tossing 4 heads in a row with a coin?

Solution: $p(\text{head}) = \tfrac{1}{2}$

$$p(h, h, h, h) = \tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2}$$
$$= \tfrac{1}{16}$$

Solve the following problems.

1. If a coin and a die are thrown, what is the probability of getting a tail and a 5?
2. What is the probability of getting a nine and a ten in successive draws from a deck of cards if the cards are replaced after each draw?
3. What is the probability of tossing 5 tails in a row with a coin?
4. The probability that Fred will win the 100-yard dash in a track meet is $\frac{1}{3}$. The probability that Jack will win the pole vault is $\frac{1}{4}$. What is the probability that both will win?
5. What is the probability that Fred will win while Jack loses in Problem 4? What is the probability that Jack will win while Fred loses?

Dependent Events

Let us consider successive acts that are dependent one on another. Suppose three people draw slips of paper from a hat. A numeral is written on each slip, and there are 25 slips in the hat. The order of drawing is Bill, Mary, and then George. If the numbers named on the slips are the integers from 1 to 25 inclusive, the probability that Bill will draw a number divisible by 5 is $\frac{5}{25}$. If Bill succeeds, there are only 4 numbers remaining out of 24 that are divisible by 5. Therefore, the probability that Mary will draw a number divisible by 5 is $\frac{4}{24}$. If Bill does *not* succeed, the probability that Mary will is $\frac{5}{24}$. Finally, George's probability of drawing a number divisible by 5 could be either $\frac{5}{23}$, $\frac{4}{23}$, or $\frac{3}{23}$ depending upon the success or failure of Bill and Mary. Since the probability of success for the participants in the drawing is dependent upon what the previous person did, the events are said to be *dependent*.

This is the Rule for solving for the probability of dependent events.

Rule 2: Suppose acts A, B, C, etc. are performed in succession. Let $p(A)$ be the probability of the success of act A, $p(B)$ be the probability of success of act B *if act A succeeds*, $p(C)$ be the probability of success of act C *if acts A and B both succeed*, etc. Then the probability of success of the successive acts is the product of the probability of the success of each act *if the previous acts succeed.*

$$p(A, B, C, \text{etc.}) = p(A) \times p(B) \times p(C) \text{ etc.}$$

EXAMPLE

What is the probability that a person will draw 3 hearts in succession from a deck of cards if the cards are not replaced after each draw?

Solution: Let H_1, H_2, and H_3 represent the act of drawing a heart on the first, second, and third drawings, respectively. Then:

$$p(H_1) = \tfrac{13}{52}; \qquad p(H_2) = \tfrac{12}{51}; \qquad p(H_3) = \tfrac{11}{50}.$$

Then $$p(H_1, H_2, H_3) = \tfrac{13}{52} \times \tfrac{12}{51} \times \tfrac{11}{50}$$
$$= \tfrac{11}{850}$$

Find the solutions to the following problems.

1. What is the probability that a person will draw 2 aces in succession from a deck of cards?

2. What is the probability that a person will draw a club, a diamond, a heart, and a spade in that order from a deck of cards?

3. Mr. Howard thoroughly shuffled a deck of cards, and the top 4 cards were all aces. What is the probability that this will happen on the next shuffle?

4. Suppose the four aces in Problem 3 were to be the ace of clubs, ace of diamonds, ace of hearts, and ace of spades in that order. What is the probability that this will happen?

5. A bag contains 5 red marbles, 7 black marbles, and 3 green marbles. What is the probability that a person will draw 3 black marbles in succession from the bag if he does not look inside while he is drawing?

6. What is the probability that a person will draw a green marble, then a black marble, and then a red marble in succession from the bag in Problem 5?

7. If a deck of cards is thoroughly shuffled, what is the probability that the first four cards dealt are spades?

8. What is the probability that the four cards dealt in Problem 7 are the ace, king, and queen of clubs in that order with the fourth club being any other club from the deck?

9. In a drawing in which there were 150 participants, Fred and Bob won first and second prize respectively. What is the probability that this will happen again if the same number participate?

10. If only 18 participate, what chance do Fred and Bob have to repeat?

11. Joe bet a dollar against one hundred dollars that he would draw two aces from a deck of cards in two tries. Was this a good bet? Why, or why not?

Permutations With Some Objects Alike

The number of permutations of the numerals 1, 2, 3, and 4, taken all at a time is ${}_4P_4$ or 24. Now, how about the permutations of the four numerals 1, 1, 1, and 4? It is plain that the interchange of the first three numerals of the first set 1, 2, 3, 4 produces a different permutation. The interchange of the first 3 numerals of the second set 1, 1, 1, 4 does *not* give a different permutation.

Therefore we must divide ${}_4P_4$ by ${}_3P_3$ to "remove" the number of arrangements of the three 1's which do not count. So we have

$$\frac{{}_4P_4}{{}_3P_3} = \frac{24}{6} = 4$$

This leads us to a formula for the number of permutations among objects some of which are alike.

Let P represent the number of different permutations of n objects taken all at a time, when of the n objects, s are alike, t are alike, u are alike, etc. Then

$$P = \frac{n!}{s! \times t! \times u!}$$

EXAMPLES

1. Find the number of permutations that can be obtained from the numerals 2, 2, 2, 3, and 4 taken all at a time.

   Solution: $$P = \frac{5!}{3!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = 20$$

2. How many different permutations can be obtained by using the letters of the word *Mississippi* taken all at a time?
   Solution: There are 11 letters in all. There are 4 i's, 4 s's, and 2 p's.

   $$P = \frac{11!}{4! \times 4! \times 2!} = 34{,}650$$

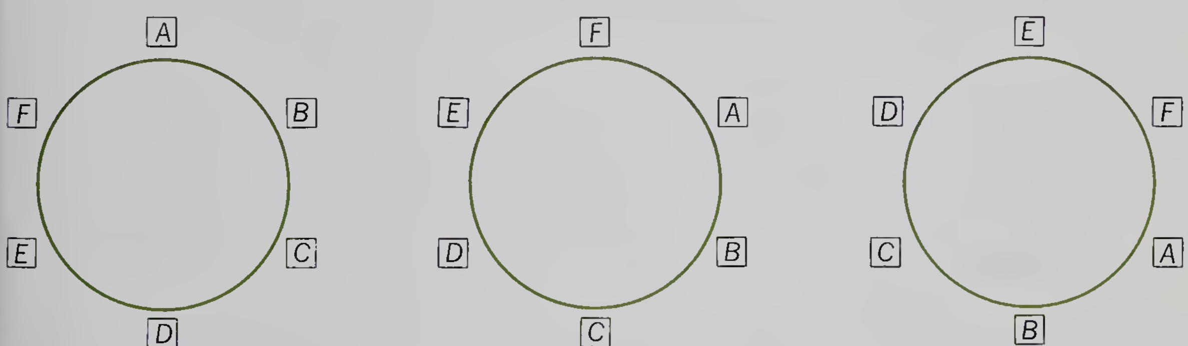
Use the Formula above to answer the following:

1. List the 20 permutations that can be obtained from the numerals 2, 2, 2, 3, and 4 taken all at once, as indicated in Example 1 above.
2. How many different numbers named by 8 digits can be formed from three 4's, two 5's, and three 6's?

3. In how many different ways can 7 identical boxes of candy and two identical bottles of perfume be distributed among 9 girls?
4. Using all the letters of the word *Illinois*, how many different permutations can be obtained?
5. In how many different ways can 5 pennies, 2 nickels, and 3 dimes be given to 10 children if each is to receive a coin?
6. How many different signals of 10 flags can be sent with 6 yellow and 4 green flags?
7. In a baseball team of 9 players, how many different batting orders can the manager select?

Circular Permutations

Let us consider the problem of arrangements of objects in a circle. We shall consider the relative order of arrangement; that is, who is on whose left and right. As an example, in how many different ways can 6 different people seat themselves around a circular table? Notice that if, seated in any order, all move one place (to the right, for example) this would not count as a different arrangement. Since they could do this six times at a different shaped table, we "lose" six ways for every seating arrangement because the table is circular. The Figure below shows three of these six ways. The letters represent people.



Then, by using the Principle on page 416, there are 5 possibilities for seat 1, 4 for seat 2, 3 for seat 3, 2 for seat 4, and 1 for seat 5. Therefore, the total number of seating arrangements is

$$5 \times 4 \times 3 \times 2 \times 1 = 120$$

This leads us to the following Formula.

Let N be the number of circular permutations of n objects taken all at a time. Then

$$N = (n - 1)!$$

EXAMPLES

1. In how many different ways can 5 people sit around a circular table?
   Solution: $N = (5 - 1)! = 4! = 4 \times 3 \times 2 \times 1 = 24$ ways

2. Four boys and four girls are to sit around a circular table with boys and girls alternating. How many different seating arrangements are there?
   Solution: First, in how many ways can the 4 boys, B, seat themselves around the table in alternate seats?

$$B = (4 - 1)! = 3!$$
$$= 3 \times 2 \times 1 = 6$$

   There are still 4 vacant seats. Using the Principle on page 416, there are $4 \times 3 \times 2 \times 1$ possible arrangements for the girls, G, in the 4 seats.

$$G = 4 \times 3 \times 2 \times 1 = 24$$

   Boys with Girls $= 6 \times 24 = 144$ ways.

Solve the following problems for circular tables.

1. In how many different ways can 7 people sit around a table?
2. In how many different ways can 5 boys and 5 girls sit around a table so boys and girls alternate?
3. Six girls are seated at a table. After they are seated, six boys stand behind the chair of each girl. How many different arrangements can there be?
   HINT: In how many different ways can the boys arrange themselves for each seating arrangement of the girls?
4. The formula for the number of different ways objects can be put on a reversible ring (one that can be turned over) is $N = \frac{(n-1)!}{2}$. This is exactly half of the number of circular permutations. Why?
5. Harry has 5 keys on a key ring. How many different key arrangements can he have?
   Note: Use the formula in Problem 4.
6. How many different ways can 5 people sit around a table if 2 people insist on sitting next to one another?
   HINT: Imagine the 2 people as one unit. However, do not forget they can trade seats and still sit next to one another.
7. In how many ways can 5 people sit around a table if 2 people are not to sit next to one another?
   HINT: Subtract the answer to Problem 6 from the total number of possible seating arrangements.

1. There were 7 books from which Mary was to pick 3 to read. How many different combinations could she choose?
2. Coach Fairchild has 6 runners from which to pick a relay team of 4 runners. How many different teams could he have picked?
3. From a group of 7 men and 5 women a committee of 5 people was to be picked. How many different committees were possible?
4. If the 5-person committee were to include *exactly* 2 women, how many committees would be possible?
5. If the committee, in Problem 3, is supposed to have *at least* 2 men, how many committees are possible?
   HINT: Add the totals with exactly 2, 3, 4, and 5 men.
6. George has a penny, a nickel, a dime, a quarter, and a 50¢ piece. How many different sums of money can he get from 3 coins? 2 coins? 4 coins?
7. How many different amounts of money can George get from the coins, in Problem 6, using all the possible combinations?
8. Mr. Williams is one of 15 men from which a committee of 4 men is to be chosen. If Mr. Williams is definitely to be on the committee, in how many ways can the other 3 positions be filled?
9. If all 4 vacancies on the committee, in Problem 8, are to be picked at random, what is the probability that Mr. Williams will be picked for the committee?
10. The probability that Mr. Jones will live 20 more years is $\frac{1}{3}$. The probability that Mrs. Jones will live 20 more years is $\frac{3}{4}$. Find the probability that both will live 20 more years. What is the probability that Mr. Jones will live 20 more years and Mrs. Jones will not? What is the probability that neither will live for 20 more years?
11. The probability that Mr. Williams will live 10 more years is $\frac{1}{4}$ and for Mr. Harrold, $\frac{1}{5}$. What is the probability that both will live for 10 more years?
12. From a deck of cards we make 2 draws, replacing the cards each time. What is the probability of drawing 2 spades?
13. What is the probability of drawing 2 cards of the same suit, if the cards are replaced each time?
14. Out of a group of 25 club members, all with an equal chance, what is the probability that Mary and Edith will be chosen as president and vice-president respectively?
15. What is the probability that Mary will be president?

Two events are said to be *mutually exclusive* if the occurrence of one event makes the second event impossible. For example, in drawing cards from a deck, in how many different ways can a person draw a club? The answer is 13 ways since there are 13 clubs in the deck. In like manner, there are 13 different ways of drawing a spade. The drawing of a club and a spade in one draw are said to be mutually exclusive events since the success in one of the events cannot be a success in the other.

An example of two events which are not mutually exclusive is drawing an ace and drawing a spade from the deck in one draw. Drawing the ace of spades is a successful event both in drawing an ace and also in drawing a spade.

Here is another Rule of probability concerning mutually exclusive events.

Rule 3: Let event A and event B be mutually exclusive events. Let the probability that A shall happen be p_A and the probability that B shall happen be p_B. Then the probability that event A or B shall happen in any trial is $p_A + p_B$. That is

$$p(A \text{ or } B) = p_A + p_B.$$

EXAMPLES

1. What is the probability of rolling either a 1 or a 6 in one roll of a die?

Solution: p_1 = probability of rolling 1
p_6 = probability of rolling 6

$$p_1 = \tfrac{1}{6}; \qquad p_6 = \tfrac{1}{6}; \qquad p(1 \text{ or } 6) = \tfrac{1}{6} + \tfrac{1}{6} = \tfrac{1}{3}$$

2. What is the probability of rolling a 6 or a 7 with a pair of dice?

Solution: From the table on page 415 $p_6 = \frac{5}{36}$; $\quad p_7 = \frac{6}{36}$

$$p(6 \text{ or } 7) = \tfrac{5}{36} + \tfrac{6}{36} = \tfrac{11}{36}$$

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1. Kenneth White, coach of the basketball team, stated that the probability of his team winning the first game was $\frac{1}{4}$ and of winning the second game was $\frac{1}{5}$. On this basis, what would be the probability that the team would win at least one of the first two games?
2. If two dice are tossed, what is the probability of either a 7 or an 11? either a 9 or an even number? a 4, a 5, or a 6?

3. If one die is tossed, what is the probability of finding either a 7 or a 5? a 2, 3, or 4? either a 7 or an even number?

4. If you toss two coins and roll one die, what is the probability of either 2 heads and an even number on the die, or 2 tails and an odd number? one head only, and either a 5 or a 3 on the die? one head only, and either a 2, a 3, or a 4 on the die?

5. Suppose you have a penny, a nickel, and a dime. The chart lists the various ways that the 3 coins may fall (heads or tails). What is the probability that the 3 coins show all heads?

Cent	*Nickel*	*Dime*
H	*H*	*H*
H	*H*	*T*
H	*T*	*H*
H	*T*	*T*
T	*H*	*H*
T	*H*	*T*
T	*T*	*H*
T	*T*	*T*

6. What is the probability that the coins will fall either all tails or all heads? two heads and one tail?

7. Add a 25¢ piece to the 3 coins in Problem 5 and fill out a chart showing all the possible combinations of heads and tails that can be obtained.

   HINT: You can do this very easily by adding 8 *H*'s and then 8 *T*'s to the 8 combinations in Problem 5, getting 16 ways in all. Why?

8. What is the probability that all 4 coins will be heads? What is the probability that there will be 3 heads and 1 tail? What is the probability that there will be 2 heads and 2 tails?

9. From the chart in Problem 7, what is the probability that at least 1 head will be tossed?

10. If three cards are drawn from a deck of cards and replaced after each draw, what is the probability of drawing a nine, a ten, or a jack in the 3 draws?

11. A jar contains 3 blue marbles, 7 red marbles, and 5 green marbles. What is the probability that we will pick either a blue or red marble from the jar if we take out 2 marbles together?

12. Bill has 5 books in his locker. What is the probability that he will select (without looking) his biology text or English text in his first try?

13. Bill rolled a die and drew a card from a deck of cards. What is the probability that he rolled a 3 on the die or drew a 3 from the deck of cards?

14. Helen said the probability that Bill would ask her to the prom was $\frac{1}{3}$ and that the probability John would ask her was $\frac{1}{2}$. On the basis of this estimate, what is the probability that either Bill or John will ask her?

The "odds in favor" or "odds against" a particular event are expressions familiar to everyone. Quoting the odds at a horse-racing track for betting purposes is an example. Suppose the odds against a certain horse winning a race are quoted at 5 to 1. This means it is thought that the horse has 5 times as many chances to lose as to win.

The use of "odds" that an event will happen, however, is not confined to gambling. Insurance companies compute their rates largely on the "odds" that a certain event will happen. Many of our everyday decisions are based upon the idea that it is more likely that one thing will happen than another.

What is the relationship between odds and probability? Remember that the probability an event will occur (p) is given by the Formula, $p = \frac{s}{t}$. In this Formula s is the number of ways the event can succeed, and t is the number of ways the event can happen.

Rule 4: If s is the number of ways an act can be successful, and t is the total number of ways the act can happen, then the odds-in-favor of a successful act are s to $(t - s)$. The odds against a successful act are $(t - s)$ to s.

EXAMPLES

1. What are the odds in favor of rolling a 4 in one cast of a die?

Solution: $s = 1$, $t = 6$. Odds in favor $= s$ to $(t - s)$, or 1 to 5.
Odds against $= (t - s)$ to s, or 5 to 1.

2. The odds against Jack winning the school election were quoted at 4 to 3. At these odds, what is the probability he will win?

Solution: $\frac{t - s}{s} = \frac{4}{3}$

Since $s = 3$, then $t - s = t - 3 = 4$. Therefore, $t = 7$

$$p = \frac{s}{t} = \frac{3}{7}$$

Use Rule 4 to answer the following:

1. What is the probability that you will draw 2 clubs in succession from a deck of cards if the cards are replaced after each draw? What are the odds against this happening?

2. The odds against the San Francisco Giants winning the National League Pennant were quoted at 3 to 1. At this rate, what is the estimated probability that they will win?
3. The odds that a race horse would lose a recent race were quoted at 5 to 2. What is the probability that the horse will win?
4. Mary said that the probability that she would go to college was $\frac{2}{5}$. What are the odds in favor of her going?
5. What are the odds against drawing 3 queens in succession from a deck of cards if the cards are replaced after each draw?
6. What are the odds against tossing a coin and getting four heads in a row?
7. What are the odds against being dealt 5 hearts in a row from a deck of cards?
8. Out of a list of 7 books, what are the odds against selecting a certain book if the choice is made at random?
9. What are the odds against rolling two successive fives in two rolls of a die?
10. What is the probability that a person will roll two successive fives in two rolls of a die?
11. The probability that Mr. Johnson will live to be eighty years old is $\frac{3}{5}$. What are the odds in favor of his reaching this age?
12. If the probability that a certain horse will win a race is $\frac{5}{6}$, what are the odds that the horse will lose?
13. What are the odds for drawing two aces in a row from a shuffled deck of cards if the cards are not replaced after each draw?
14. Mary said that the probability that she would become a secretary was $\frac{4}{7}$. What are the odds that she will not do this type of work?
15. If the probability that the Giants will win the pennant is $\frac{2}{3}$, what are the odds that they are successful?
16. Fred said that the odds in favor of his passing a math test were 3 to 1. What is the probability that he will fail?
17. What are the odds against tossing three fours in a row, in tossing a single die?
18. If it is going to rain three days out of a given week, what are the odds that it will rain on Sunday?
19. The probability that Mr. Smith will live to be seventy years old is $\frac{2}{3}$. The probability that Mrs. Jones will live to be seventy is $\frac{1}{3}$. What are the odds in favor of both reaching the age of seventy years?
20. What are the odds on a "fifty-fifty" chance?

SOME THINGS TO REMEMBER

Remember that "probability is probability." If a weather forecaster says that the probability of rain is $\frac{5}{6}$ for 5 days in a row, it still might not rain in any of the 5 days. It might be that the data he based his predictions on were faulty. It also might be that an exceptional situation arose where the data indicated a strong probability that it would rain, but for some unexplained reason, it did not.

Suppose a coin is tossed 10 times. Our probability rules tell us that we will probably get heads and tails each half the time. However, is it not possible that we can get heads in all 10 tosses? This situation, of course, is quite exceptional. The odds against it happening are 1023 to 1, but it can happen.

Each of the four rules studied in this chapter is based upon the idea that each of the events has an equal chance of happening. Naturally, if the dice are not perfectly balanced so that each face has an equal chance of coming up, the probabilities will not be correct. Similarly, if the deck of cards is prearranged in favor of a certain draw, we cannot use the rules to compute the probability. If a die is weighted to favor a certain number, it is said to be loaded. One can approximate the probability that each number will come up only by experimentation. However, to be accurate, the experimentation must involve many trials. Observed probability of this type is called *empirical* or *experimental* probability.

In everyday life, the stated probability of a certain event is usually based upon analysis and interpretation of data related to the event. If the analysis and interpretation are at fault, the probability that a stated event will occur will also be distorted.

The accumulation of valid data and their proper evaluation and interpretation are still as important as anything else. The principles of probability introduced in this chapter are useful in making the best possible decisions and conclusions in many fields of endeavor. Science, business, government, education, and others use these ideas and principles constantly to aid in making valid predictions and decisions.

All too often, when people apply the rules concerning probability, if the rules fail them, they tend to discount the accuracy and validity of the mathematical analysis. However, the mathematics will prove to be correct in a sufficient number of trials. Also, if the mathematical analysis seems to be incorrect, it may very well be that the data are either insufficient, faulty, or improperly applied. In spite of its rather shady beginning among the gambling elements, and with its numerous applications in gambling games, the study of probability and chance has achieved a high level of respectability in mathematics.

Part One

A. List the numerals 1 to 7 on a sheet of paper. After each numeral write T or F, to indicate whether each of the following statements is true or false.

1. The probability of an impossible event happening is zero.
2. The statement "a fifty-fifty chance" means a probability of $\frac{1}{50}$.
3. A permutation does not take into account the order of arrangement.
4. The probability of drawing a spade from a deck of cards is greater than the probability of drawing a king, queen, or jack.
5. The letters in the word "five" can be arranged in exactly 24 different ways.
6. A coin is tossed and comes up heads. On the next toss it is more likely to come up tails.
7. There are more permutations of 5 objects taken 5 at a time than there are combinations of 5 objects taken 5 at a time.

B. Answer the following questions.

1. What is the probability that a person in tossing a coin 5 times will toss 5 heads?
2. In throwing a die, what is the probability that a number will come up greater than 2 and less than 5?
3. What is the probability of drawing 3 diamonds in succession from a deck of cards if the cards are replaced after each draw?
4. What is the probability when rolling two dice, that a number greater than 5 will appear on at least one die?
5. Three black balls and 7 white balls are in a bag. What is the probability that a person will remove a black ball and then a white ball in successive draws if the ball is returned after the first draw?
6. What is the probability that a person will draw 5 cards of the same suit in succession from a deck of cards if the cards are not replaced after each draw?
7. What is the probability of drawing the four aces in succession from a deck of cards?
8. In throwing a die, what is the probability that the number which comes up is even?
9. What is the probability, if 3 coins are tossed, that all 3 will come up tails?

Part Two

1. Carol wanted to rearrange 7 books on a shelf. In how many different ways can the 7 books be arranged?
2. Using only the digits 1, 2, 3, 4, and 5, how many different 3-digit numerals can be formed?
3. Using the same digits as in Exercise 2, how many different 3-digit numerals can be made if no digit is to be repeated?
4. In how many ways can 5 girls and 4 boys occupy 9 seats in a row if boys and girls are to occupy alternate seats?
5. From 5 Spanish books, 4 German books, and 3 French books, how many sets of 6 books, each set with 2 books of each language, can be obtained?
6. Four boys and three girls are to sit in a row of seven chairs. If the girls are to sit together, how many different seating arrangements can there be?
7. How many different batting orders can a baseball coach get using 9 players if the pitcher must bat ninth?
8. In how many ways can the position of President, Secretary, and Treasurer be filled from a club of 14 members if no member can hold 2 offices?
9. How many different committees of 3 members can be formed from a club of 14 members?
10. How many different signals can be sent from 5 signal flags, each of different color, if a signal consists of 4 flags arranged vertically?
11. Coach Wiser has 12 basketball players from which to choose a starting five. How many different combinations of 5 players can he choose?
12. Mary had 4 pairs of shoes, three skirts, and five sweaters. How many different outfits might she choose from the shoes, skirts, and sweaters?
13. Coach Fairchild has 12 players from which to fill the four backfield positions of the football team. How many different backfield line ups can he choose?

Part Three

1. A committee of 4 is to be selected from a group of 9 people. How many possible committees are there?
2. Fred has 7 books from which to choose 3 for a reading assignment in his English class. How many different choices does he have?

3. From a group of 6 men and 5 women, how many committees can be chosen with exactly 3 men and 3 women on each?
4. The track coach at Middleton High School had 11 runners from which to pick a 4-man relay team. How many different teams could he pick?
5. Find the number of different 13-card bridge hands that can be dealt from a deck of 52 playing cards.
6. The annual school play has parts for 2 boys and 3 girls. Five boys and 8 girls have tried out. How many different casts for the play can be chosen?
7. From a list of 5 language and 5 science books, George wishes to choose 4 books. At least 2 of the books must be science books. How many different choices can he make?
8. In how many ways can a club of 12 members form a 4-member committee if the president is always a member of each committee?
9. How many different sums can be obtained from a penny, a nickel, and a dime?
10. A biology test contained 8 questions. The students were instructed to choose any 5 of the 8 questions. How many different choices did the students have?
11. From a list of 8 different colors, Mrs. Haskins wishes to choose 3 for a color scheme for her living room. How many color schemes can she choose?
12. From a class of 22 students, 5 were to be chosen to go on a field trip. How many different groups of 5 can be selected?
13. From a group of 28 astronauts, three are to be chosen for a trip to the moon. How many possible crews for the moon trip are there?
14. The school debate team consists of 4 members. There are 7 active members of the debate club. How many different teams can be chosen from the 7 active members of the club?
15. If 4 members of the debate club are boys and 3 are girls, how many different debate teams consisting of 2 boys and 1 girl are possible?
16. A baseball coach has 4 pitchers plus 12 other players from which to choose a line up. How many different line ups might he choose if the pitchers must do the pitching and play no other position?
17. The letters of the alphabet are written on slips of paper and placed in a hat. As the slips are drawn, the letter is written on the board and the slips returned to the hat. What is the probability that the first 6 letters written will spell MURPHY in that order?

Part One

A. Write each of the following as a decimal rounded to the nearest thousandth, where necessary.

1. $\frac{1}{4}$	**3.** $\frac{2}{5}$	**5.** $\frac{7}{13}$	**7.** $\frac{2}{15}$
2. $\frac{9}{10}$	**4.** $\frac{5}{8}$	**6.** $\frac{7}{9}$	**8.** $\frac{7}{8}$

B. Write each of the following as a per cent.

1. 0.63	**3.** 0.07	**5.** 0.024	**7.** 2.43
2. 0.15	**4.** 0.265	**6.** 0.005	**8.** 6.5

C. Write each of the following as a decimal.

1. 65%	**3.** 2.5%	**5.** 12.5%	**7.** 9.8%
2. 46.2%	**4.** 0.6%	**6.** 216.4%	**8.** 350%

Part Two

In the column on the right below is a list of words used in the previous chapters, and on the left are 10 definitions. Write the numerals 1 through 10 on a sheet of paper. After each numeral write the word defined by the definition having that numeral.

1. A per cent of the sales received by the salesman as earnings	commission
2. The balance of the margin after overhead has been paid out	date of maturity
3. The date on which a note is due	face
4. A signed agreement to repay a loan	interest
5. Income paid to the lender by the borrower	maker
6. Signer of a promissory note	margin
7. Length of time money is borrowed	overhead
8. Difference between cost and selling price	principal
9. The sum on which the borrower pays interest	profit
10. Costs of doing business	promissory note
	rate
	selling price
	term

Part Three

Find the value for N in each of the following:

1. 62.5% of 96 is N

2. N% of 36 is 80

3. 0.52% of 650 is N
4. 7.5% of 96 is N
5. N is 175% of 32
6. 125% of 560 is N
7. 60% of N is 15
8. 5% of N is 30
9. 2.5% of N is 16
10. 0.8% of N is 48

Part Four

Find the missing number.

	Principal	*Term*	*Rate*	*Interest*
1.	\$800	90 days	5%	?
2.	?	72 days	6%	\$24
3.	\$175	?	4%	\$3.50
4.	\$375	18 months	?	\$22.50
5.	\$600	75 days	$4\frac{1}{2}\%$	?
6.	\$325	90 days	3%	?

Part Five

1. How many different sums of money can be made up with a penny, a nickel, and a dime, using one, two, or three coins at a time?
2. Henry's father owns an orchard. Henry gets a 15% commission on all the fruit he sells. Last week Henry sold 250 pounds of pears at 20¢ a pound. How much was his commission?
3. A rug regularly priced at \$375 is sold for \$225. What is the per cent of reduction from the regular price?
4. By purchasing a book at a "25% off" sale, Elizabeth saved 80¢. What did she pay for the book?
5. In how many ways can 4 people be selected from a group of 7 people?
6. A salesman at the Sport Shop earns a commission of 12% on his sales. Last week his commission was \$180. How much were his sales?
7. Four books are to be placed on a shelf. In how many ways can they be arranged?
8. If a die is rolled, what is the probability of an even number turning up?
9. Mr. Hall purchases cat food at \$2.88 per case of 24 cans. He sells it at 20¢ a can. The margin is what per cent of the selling price?
10. Two balls are drawn from a box containing 3 red balls and 2 white balls. What is the probability that both balls drawn will be red?

EXTRA PRACTICE

The practice units in the pages that follow are keyed to the Inventory Tests in Chapter 4 and following, where these Tests are not followed by PRACTICE EXERCISES. Throughout the text the Inventory Tests are designed to identify the special kinds of computations on which you need practice. It is to these areas of difficulty that you should devote your attention. The Practice Exercises are keyed to these tests so that your practice will be directed to correcting your special difficulty.

Thus, for example, *C* of the Inventory Test on fractions in Chapter 4 covers addition with fractions. Therefore, *C* of the Practice Exercises on page 446 in the following section provides exercises on addition of fractions. If your test shows that you need practice, these are the exercises for you to use.

The *EXAMPLES* included with the Practice Exercises show you how the operations should be performed, in case you have forgotten. Be sure to study the *EXAMPLES* and to practice using the correct procedures. It is as important to practice correct procedures as it is to get the right answers. You should strive to develop speed after you have developed accuracy, not before.

Do not waste time copying any exercises except when it is necessary. In addition, subtraction, and multiplication of whole numbers and numbers named by decimals, you can usually place a sheet of paper below the exercise, and write only the answer. After you have done the first row, you can fold the paper and do the second row, and so on.

Each set of Practice Exercises is followed by a Practice Test that will help you find out if you have mastered the computations. When you have completed work on the Practice Exercises, your teacher will give you directions for taking the Practice Test. When this test has been scored, examine those exercises that you missed to determine the causes of your errors.

Remember! Practice leads to improvement only if you are learning correct procedures. Usually, the correct procedures you need to learn are those illustrated in the *EXAMPLES*. If so, you can readily learn them by practicing them. On the other hand, your difficulty may be due not to incorrect procedures, but rather to failure to check your answers, to write legible figures, or to carelessness in arranging your work. A little time spent in arranging your work neatly, writing the numerals legibly, and in checking your work can pay big dividends in saving time by avoiding mistakes.

A. *Simplest form*

Rule: To express fractions in their simplest form, divide the numerator and denominator by their greatest common factor.

Study the Examples below.

EXAMPLES

1. $\frac{20}{36} = \frac{4 \times 5}{4 \times 9}$
$= \frac{4}{4} \times \frac{5}{9}$
$= 1 \times \frac{5}{9} = \frac{5}{9}$

What number did we divide the numerator and denominator by? Therefore, 4 is the *greatest* common factor of 20 and 36. Note that 2, of course, is a common factor, but 4 is the *greatest* common factor.

2. $\frac{\overset{5}{\cancel{20}}}{\underset{9}{\cancel{36}}} = \frac{5}{9}$

Express each of the following in simplest form.

1. $\frac{15}{25}$ **3.** $\frac{18}{30}$ **5.** $\frac{12}{48}$ **7.** $\frac{8}{36}$ **9.** $\frac{5}{40}$

2. $\frac{32}{44}$ **4.** $\frac{21}{42}$ **6.** $\frac{9}{33}$ **8.** $\frac{6}{16}$ **10.** $\frac{19}{38}$

B. *L.C.M.*

Rule: To find the L.C.M. (least common multiple), write the prime factors of each denominator. Then use each prime factor the greatest number of times it occurs in any one of the denominators as factors for the L.C.M.

Study the Example below.

EXAMPLE

Denominators: 6, 9, 4
Prime factors: $6 = 2 \times 3$; $9 = 3 \times 3$; $4 = 2 \times 2$
L.C.M.: $3 \times 3 \times 2 \times 2 = 36$

Find the L.C.M. of each of the following:

1. 4, 7 **3.** 8, 18 **5.** 3, 5 **7.** 6, 3, 4 **9.** 3, 9, 12

2. 6, 21 **4.** 9, 15 **6.** 8, 12 **8.** 8, 6, 4 **10.** 5, 10, 15

C. *Addition*

Study the Example and complete the statements. The greatest denominator, 9, is not divisible by each of the others, since it is not divisible by ? or ?. Multiplying 9 by 2 gives us ? which is not divisible by ?. Multiplying 9 by 3 gives us ? which is not divisible by ? or ?. Multiplying 9 by 4 gives us 36, which is a common denominator because ?.

EXAMPLE

$$\frac{3}{4} = \frac{?}{36}$$
$$\frac{1}{3} = \frac{?}{36}$$
$$\frac{5}{6} = \frac{?}{36}$$
$$\frac{7}{9} = \frac{?}{36}$$

Add: Write all answers in simplest form.

1. $\frac{3}{8} + \frac{1}{6} + \frac{5}{16}$

2. $\frac{2}{3} + \frac{1}{4} + \frac{5}{8}$

3. $\frac{5}{9} + \frac{3}{4} + \frac{2}{3}$

4. $\frac{5}{8} + \frac{5}{6} + \frac{11}{16}$

5. $\frac{3}{7} + \frac{1}{2} + \frac{5}{21}$

6. $\frac{3}{5} + \frac{1}{2} + \frac{3}{4}$

7. $\frac{7}{8} + \frac{3}{4} + \frac{5}{16}$

8. $\frac{5}{6} + \frac{7}{18} + \frac{1}{4}$

9. $\frac{3}{8} + \frac{3}{4} + \frac{2}{3}$

10. $\begin{array}{r} 9\frac{1}{3} \\ 3\frac{3}{10} \\ \hline \end{array}$

11. $\begin{array}{r} 1\frac{1}{4} \\ 5\frac{1}{5} \\ \hline \end{array}$

12. $\begin{array}{r} 6\frac{2}{5} \\ 3\frac{2}{7} \\ \hline \end{array}$

13. $\begin{array}{r} 11\frac{3}{8} \\ 6\frac{3}{5} \\ \hline \end{array}$

14. $\begin{array}{r} 2\frac{1}{2} \\ 3\frac{3}{14} \\ 1\frac{2}{7} \\ 2\frac{3}{7} \\ \hline \end{array}$

15. $\begin{array}{r} 7\frac{3}{8} \\ 6\frac{1}{8} \\ \frac{7}{16} \\ 1\frac{1}{2} \\ \hline \end{array}$

16. $\begin{array}{r} 4\frac{1}{3} \\ 3\frac{1}{2} \\ 2 \\ 4\frac{2}{3} \\ \hline \end{array}$

17. $\begin{array}{r} 5\frac{1}{2} \\ 7 \\ 6\frac{3}{8} \\ 3\frac{3}{4} \\ \hline \end{array}$

D. *Subtraction*

Study the Example and complete these statements: The greatest denominator, 5, is not divisible by ?. Multiplying 5 by 3 gives ? which is a common denominator because ?. Complete the subtraction.

EXAMPLE

Subtract: $4\frac{3}{5} - 3\frac{1}{3}$

$$\begin{array}{rcr} 4\frac{3}{5} & = & 4\frac{9}{15} \\ 3\frac{1}{3} & = & 3\frac{5}{15} \\ \hline & & ? \end{array}$$

Subtract: Write all answers in simplest form.

1. $\frac{5}{8} - \frac{5}{12}$

2. $\frac{8}{9} - \frac{5}{6}$

3. $\frac{3}{4} - \frac{1}{3}$

4. $\frac{9}{16} - \frac{1}{3}$

5. $\frac{4}{5} - \frac{3}{4}$

6. $\frac{5}{6} - \frac{5}{8}$

7. $\frac{8}{9} - \frac{3}{4}$

8. $4\frac{1}{6} - 2\frac{3}{4}$

9. $9\frac{7}{12} - 6\frac{5}{8}$

10. $3\frac{5}{9} - 2\frac{3}{4}$

11. $7\frac{1}{3} - 4\frac{1}{2}$

12. $5\frac{7}{9} - 3\frac{5}{6}$

13. $4\frac{1}{6} - 1\frac{1}{8}$

14. $5\frac{1}{2} - 3\frac{1}{8}$

15. $7\frac{3}{4} - 4\frac{2}{3}$

16. $13\frac{9}{16} - 7\frac{7}{8}$

17. $5\frac{5}{6} - 4\frac{7}{8}$

18. $8\frac{1}{4} - 5\frac{5}{6}$

A. *Add:* Write all answers in simplest form.

1. $\frac{3}{4}$
 $\frac{7}{8}$
 $\frac{1}{2}$

2. $\frac{5}{6}$
 $\frac{3}{4}$
 $\frac{4}{5}$

3. $3\frac{3}{5}$
 $4\frac{5}{6}$
 $3\frac{7}{8}$

4. $7\frac{4}{5}$
 $8\frac{3}{4}$
 $6\frac{3}{5}$

5. $10\frac{3}{4}$
 $4\frac{1}{12}$
 $2\frac{5}{6}$
 $11\frac{11}{16}$

6. $14\frac{7}{8}$
 $3\frac{7}{12}$
 $15\frac{2}{3}$
 $13\frac{3}{4}$

7. $14\frac{5}{6}$
 $10\frac{1}{3}$
 $8\frac{5}{18}$
 $13\frac{5}{12}$

8. $3\frac{3}{4}$
 $6\frac{1}{3}$
 $10\frac{3}{8}$
 $36\frac{3}{16}$

9. $11\frac{7}{12}$
 $6\frac{3}{5}$
 $13\frac{3}{4}$
 $19\frac{2}{3}$

B. *Subtract:* Write all answers in simplest form.

1. $\frac{5}{8}$
 $\frac{1}{2}$

2. $2\frac{8}{9}$
 $1\frac{5}{6}$

3. $3\frac{3}{5}$
 $\frac{15}{16}$

4. 8
 $4\frac{3}{5}$

5. $17\frac{3}{8}$
 $6\frac{3}{4}$

6. $23\frac{1}{2}$
 $14\frac{5}{8}$

7. $25\frac{3}{8}$
 $16\frac{1}{2}$

8. $11\frac{5}{6}$
 $3\frac{7}{9}$

9. $32\frac{5}{6}$
 $27\frac{3}{4}$

10. $33\frac{5}{9}$
 $27\frac{1}{2}$

11. $17\frac{1}{6}$
 $9\frac{1}{4}$

12. $19\frac{2}{3}$
 $11\frac{3}{4}$

13. $15\frac{7}{12}$
 $7\frac{8}{9}$

14. $31\frac{1}{2}$
 $25\frac{3}{5}$

A. *Multiplying fractional numbers*

To find the product of two fractional numbers, we find the product of the numerators and the product of the denominators. Study the procedure in the following Examples for writing the answer in simplest form.

EXAMPLES

1. $\frac{3}{7} \times \frac{1}{6} = \frac{3}{42}$

$$= \frac{3}{3 \times 2 \times 7}$$

$$= \frac{3}{3} \times \frac{1}{2 \times 7}$$

$$= 1 \times \frac{1}{14}$$

$$= \frac{1}{14}$$

2. Let us simplify the procedure.

$$\frac{3}{7} \times \frac{1}{6} = ?$$

$$\frac{\overset{1}{\cancel{3}}}{7} \times \frac{1}{\underset{2}{\cancel{6}}} = \frac{1}{14}$$

3 and 6 are both divisible by ?

Multiply: Write all answers in simplest form.

1. $\frac{7}{8} \times \frac{4}{5}$
2. $\frac{3}{4} \times \frac{8}{9}$
3. $\frac{3}{4} \times \frac{6}{7}$
4. $\frac{15}{16} \times \frac{2}{5}$
5. $\frac{3}{5} \times \frac{5}{8}$
6. $\frac{5}{9} \times \frac{18}{25}$
7. $\frac{7}{15} \times \frac{5}{7}$
8. $\frac{7}{12} \times \frac{3}{5}$
9. $\frac{15}{16} \times \frac{4}{5}$
10. $\frac{11}{15} \times \frac{5}{6}$
11. $\frac{9}{16} \times \frac{4}{9}$
12. $\frac{5}{6} \times \frac{14}{15}$
13. $\frac{5}{8} \times \frac{4}{5}$
14. $\frac{3}{16} \times \frac{4}{9}$
15. $\frac{4}{5} \times \frac{5}{7}$
16. $\frac{2}{3} \times \frac{7}{8}$
17. $\frac{5}{8} \times \frac{11}{12}$
18. $\frac{9}{14} \times \frac{7}{18}$
19. $\frac{4}{7} \times \frac{5}{8}$
20. $\frac{8}{9} \times \frac{3}{7}$
21. $\frac{2}{5} \times \frac{5}{8}$

B. *Multiplying a whole number and a fractional number*

If you have any trouble, you may think of 12 as $\frac{12}{1}$. It may be useful to write it that way.

To find the numerator in the product you find the product of what two numbers? How do you find the denominator?

EXAMPLE

Multiply $12 \times \frac{4}{5}$

$\frac{12}{1} \times \frac{4}{5} = \frac{48}{5} = 9\frac{3}{5}$

Multiply: Write all answers in simplest form.

1. $3 \times \frac{7}{8}$
2. $15 \times \frac{2}{3}$
3. $4 \times \frac{3}{8}$
4. $9 \times \frac{2}{3}$
5. $8 \times \frac{5}{6}$
6. $18 \times \frac{5}{9}$
7. $\frac{2}{5} \times \$15$
8. $\frac{3}{4} \times \$16$
9. $\frac{3}{4} \times \$18$
10. $\frac{2}{7} \times \$10.50$
11. $\frac{4}{5} \times \$12$
12. $\frac{3}{5} \times \$37.50$

C. *Multiplying a whole number by a number named by a mixed numeral*

You may first write both numbers named as improper fractions, and then proceed. However, it is often simpler to use the process shown in the Example at the right.

EXAMPLE

Find $18 \times 6\frac{2}{3}$

$$\begin{array}{rl} 18 & \\ 6\frac{2}{3} & \\ \hline 12 & (18 \times \frac{2}{3}) \\ 108 & \\ \hline 120 & \end{array}$$

Multiply: Write all answers in simplest form.

1. $36 \times 4\frac{2}{3}$
2. $9 \times 13\frac{2}{3}$
3. $8\frac{1}{4} \times 24$
4. $3\frac{3}{16} \times 18$
5. $14\frac{3}{16} \times \$36$
6. $16\frac{3}{8} \times \$20$
7. $18\frac{3}{5} \times \$30$
8. $5\frac{4}{5} \times \$6.25$
9. $68 \times 3\frac{3}{4}$
10. $72 \times 6\frac{3}{4}$
11. $6\frac{2}{3} \times 18$
12. $15 \times 4\frac{2}{3}$
13. $19\frac{3}{8} \times \$16$
14. $3\frac{3}{5} \times \$25$
15. $6\frac{7}{8} \times \$56$
16. $6\frac{2}{3} \times \$18.75$
17. $84 \times 1\frac{5}{7}$
18. $16\frac{3}{4} \times 36$

D. *Mixed numerals*

Study the example. What improper fraction is equivalent to $3\frac{3}{4}$? What improper fraction is equivalent to $2\frac{4}{5}$?

EXAMPLE

Find $3\frac{3}{4} \times 2\frac{4}{5}$

$\frac{15}{4} \times \frac{14}{5} = ?$

Can the multiplication be simplified? Complete the exercise.

Multiply: Write all answers in simplest form.

1. $14 \times 6\frac{2}{3}$
2. $4\frac{2}{5} \times \frac{3}{10}$
3. $7\frac{2}{5} \times 5\frac{1}{4}$
4. $25 \times 3\frac{1}{2}$
5. $3\frac{2}{5} \times 7\frac{1}{3}$
6. $3\frac{2}{3} \times 4\frac{2}{5}$
7. $1\frac{7}{10} \times 4\frac{1}{8}$
8. $\frac{3}{8} \times \frac{4}{5}$
9. $\frac{7}{8} \times \frac{4}{9}$
10. $\frac{5}{8} \times \frac{4}{15}$
11. $\frac{3}{4} \times \frac{5}{9}$
12. $\frac{2}{3} \times \frac{6}{7}$
13. $\frac{3}{4} \times \frac{5}{8}$
14. $2\frac{1}{4} \times 2\frac{5}{8}$
15. $6\frac{1}{2} \times 7\frac{2}{3}$
16. $6\frac{1}{6} \times 4\frac{1}{8}$
17. $11\frac{1}{2} \times 9\frac{3}{5}$
18. $3\frac{11}{12} \times 12$
19. $23\frac{1}{2} \times 2\frac{4}{5}$
20. $3\frac{3}{10} \times 6\frac{7}{8}$
21. $\frac{7}{8} \times \frac{8}{9}$
22. $\frac{5}{12} \times \frac{3}{8}$
23. $8 \times \frac{3}{4}$
24. $16 \times \frac{5}{8}$
25. $18 \times \frac{2}{9}$
26. $14 \times \frac{5}{7}$
27. $5\frac{2}{3} \times 4\frac{3}{4}$
28. $3\frac{1}{6} \times 8\frac{7}{8}$
29. $1\frac{1}{8} \times 10\frac{5}{8}$
30. $2\frac{3}{16} \times 4\frac{4}{5}$
31. $13\frac{1}{10} \times 6\frac{4}{5}$
32. $3\frac{5}{8} \times 4\frac{1}{6}$
33. $5\frac{1}{2} \times 14\frac{2}{3}$
34. $35 \times \frac{4}{5}$
35. $3\frac{1}{2} \times 4\frac{2}{7}$
36. $5\frac{5}{6} \times 2\frac{4}{5}$
37. $4\frac{2}{5} \times 3\frac{3}{4}$
38. $6\frac{1}{5} \times 2\frac{1}{2}$
39. $17\frac{1}{2} \times 4\frac{2}{5}$
40. $13\frac{5}{8} \times 12\frac{7}{10}$
41. $3\frac{5}{6} \times 7\frac{5}{6}$
42. $3\frac{5}{12} \times 4\frac{15}{16}$

43. $15\frac{1}{4} \times 17\frac{1}{5}$
44. $4\frac{3}{5} \times 3\frac{7}{8}$
45. $11\frac{1}{10} \times 3\frac{2}{3}$
46. $6\frac{5}{12} \times 9\frac{3}{4}$
47. $4\frac{1}{5} \times 3\frac{3}{4}$
48. $12 \times 6\frac{3}{4}$
49. $6 \times 4\frac{2}{3}$
50. $25 \times 3\frac{3}{5}$
51. $6\frac{3}{4} \times 24$

E. *Division by a fractional number*

For each Example below explain:
(a) What is the divisor?
(b) What is its multiplicative inverse?
(c) How was the multiplication simplified?

EXAMPLES

1. $36 \div \frac{4}{5}$

$$\frac{\overset{9}{\cancel{36}}}{1} \times \frac{5}{\underset{1}{\cancel{4}}} = ?$$

2. $\frac{8}{9} \div \frac{2}{3}$

$$\frac{\overset{4}{\cancel{8}}}{\underset{3}{\cancel{9}}} \times \frac{\overset{1}{\cancel{3}}}{\underset{1}{\cancel{2}}} = ?$$

Divide: Write all answers in simplest form.

1. $18 \div \frac{3}{4}$
2. $12 \div \frac{6}{7}$
3. $5\frac{2}{5} \div \frac{3}{5}$
4. $16\frac{2}{3} \div \frac{5}{6}$
5. $\frac{5}{8} \div \frac{5}{16}$
6. $\frac{3}{4} \div \frac{5}{12}$
7. $8\frac{1}{3} \div \frac{5}{9}$
8. $33\frac{1}{3} \div \frac{10}{11}$
9. $37\frac{1}{2} \div \frac{5}{6}$
10. $\frac{7}{12} \div \frac{5}{8}$
11. $14\frac{2}{7} \div \frac{25}{28}$
12. $\frac{15}{16} \div \frac{9}{10}$
13. $25 \div \frac{5}{9}$
14. $14 \div \frac{7}{8}$
15. $\frac{5}{12} \div \frac{5}{6}$

F. *Dividing a fractional number by a whole number*

Study the Example. What is the divisor? What is its multiplicative inverse? Copy and complete the exercise.

EXAMPLE

$\frac{8}{15} \div 14$

$\frac{8}{15} \div \frac{14}{1}$

$\frac{8}{15} \times \frac{1}{14} = ?$

Divide: Write all answers in simplest form.

1. $\frac{4}{5} \div 8$
2. $\frac{16}{25} \div 4$
3. $\frac{5}{6} \div 20$
4. $\frac{8}{15} \div 12$
5. $\frac{3}{4} \div 9$
6. $1\frac{1}{4} \div 5$
7. $6\frac{1}{4} \div 20$
8. $7\frac{1}{8} \div 19$
9. $6\frac{7}{8} \div 5$
10. $5\frac{1}{4} \div 14$
11. $\frac{2}{5} \div 4$
12. $\frac{5}{8} \div 5$
13. $\frac{3}{16} \div 6$
14. $\frac{14}{15} \div 7$
15. $\frac{18}{25} \div 9$
16. $\frac{9}{10} \div 3$
17. $\frac{6}{7} \div 4$
18. $\frac{15}{16} \div 5$
19. $\frac{24}{25} \div 8$
20. $\frac{16}{17} \div 4$
21. $2\frac{1}{2} \div 5$
22. $3\frac{3}{7} \div 8$
23. $8\frac{2}{5} \div 7$
24. $7\frac{1}{2} \div 3$
25. $8\frac{1}{3} \div 5$
26. $9\frac{3}{8} \div 25$
27. $6\frac{3}{7} \div 9$
28. $4\frac{1}{2} \div 3$
29. $5\frac{7}{16} \div 29$
30. $11\frac{1}{9} \div 25$

G. *Dividing by numbers named by mixed numerals*

Study the Examples. How were mixed numerals treated? Name the divisor. What is its multiplicative inverse? Copy and complete the exercise.

EXAMPLES

1. $3\frac{3}{4} \div \frac{5}{6}$
   $\frac{15}{4} \times \frac{6}{5} = ?$
2. $6\frac{2}{3} \div 8\frac{1}{3}$
   $\frac{20}{3} \times \frac{3}{25} = ?$

Divide: Write all answers in simplest form.

1. $3\frac{3}{5} \div \frac{9}{10}$
2. $5\frac{5}{8} \div \frac{3}{5}$
3. $6\frac{1}{4} \div \frac{15}{16}$
4. $2\frac{1}{3} \div \frac{7}{8}$
5. $7\frac{3}{5} \div \frac{14}{15}$
6. $5\frac{5}{6} \div 5\frac{1}{11}$
7. $7\frac{1}{2} \div 3\frac{3}{4}$
8. $8\frac{1}{4} \div 5\frac{1}{2}$
9. $4\frac{1}{2} \div 1\frac{7}{8}$
10. $7\frac{1}{2} \div 1\frac{1}{4}$
11. $5\frac{1}{8} \div 5\frac{1}{2}$
12. $3\frac{3}{4} \div 1\frac{7}{8}$
13. $5\frac{3}{8} \div 2\frac{3}{4}$
14. $6\frac{3}{8} \div 2\frac{5}{6}$
15. $1\frac{3}{4} \div 12$

Divide: Write all answers in simplest form.

16. $5\frac{3}{5} \div 8$
17. $3\frac{1}{3} \div 10$
18. $5\frac{1}{3} \div 9$
19. $4\frac{1}{8} \div 11$
20. $2\frac{2}{7} \div 12$
21. $3\frac{3}{5} \div 15$
22. $29\frac{5}{8} \div 79$
23. $12\frac{3}{4} \div 25$
24. $156\frac{1}{4} \div 25$
25. $31\frac{3}{4} \div 65$
26. $15\frac{5}{9} \div \frac{5}{6}$
27. $3\frac{3}{5} \div \frac{6}{7}$
28. $18\frac{2}{5} \div \frac{5}{6}$
29. $2\frac{1}{7} \div \frac{5}{8}$
30. $2\frac{1}{3} \div \frac{7}{8}$
31. $\frac{31}{40} \div 1\frac{7}{16}$
32. $3\frac{9}{11} \div \frac{6}{7}$
33. $\frac{14}{15} \div 1\frac{2}{5}$
34. $\frac{15}{16} \div 1\frac{1}{8}$
35. $21\frac{1}{3} \div \frac{8}{9}$
36. $1\frac{5}{9} \div 3\frac{1}{5}$
37. $2\frac{1}{2} \div 2\frac{1}{7}$
38. $2\frac{1}{8} \div 1\frac{17}{64}$
39. $6\frac{2}{5} \div 1\frac{3}{7}$
40. $7\frac{1}{2} \div 6\frac{1}{4}$
41. $71\frac{3}{22} \div 142\frac{3}{11}$
42. $15\frac{7}{8} \div 16\frac{1}{4}$
43. $5\frac{5}{7} \div 2\frac{3}{5}$
44. $50\frac{3}{4} \div 2\frac{1}{5}$
45. $70\frac{5}{6} \div 2\frac{1}{12}$

PRACTICE TEST

A. *Multiply:* Write all answers in simplest form.

1. $\frac{7}{8} \times \frac{4}{5}$
2. $\frac{3}{5} \times \frac{5}{16}$
3. $\frac{7}{8} \times 3\frac{1}{5}$
4. $5\frac{5}{6} \times \frac{3}{5}$
5. $3\frac{3}{8} \times 5\frac{1}{3}$
6. $3\frac{3}{7} \times 35$
7. $6\frac{5}{9} \times 18$
8. $2\frac{2}{5} \times 3\frac{4}{7}$
9. $\frac{5}{9} \times 3\frac{1}{5}$

B. *Divide:* Write all answers in simplest form.

1. $\frac{8}{15} \div \frac{2}{3}$
2. $\frac{5}{8} \div \frac{3}{4}$
3. $3\frac{1}{3} \div \frac{5}{6}$
4. $14\frac{2}{7} \div 16\frac{2}{3}$
5. $7\frac{1}{5} \div 5\frac{1}{7}$
6. $25 \div 6\frac{2}{3}$
7. $6 \div \frac{3}{8}$
8. $6\frac{2}{3} \div 15$
9. $\frac{12}{25} \div 4$
10. $\frac{15}{16} \div \frac{5}{24}$
11. $4\frac{1}{2} \div \frac{9}{10}$
12. $6\frac{2}{3} \div 1\frac{1}{3}$

A. *Making columns even*

Sometimes it is necessary to add numbers of tenths, hundredths, and thousandths, named in a column so the right-hand side is uneven. If the numerals do not represent weight, length, or some other measurement, *zeros* may be annexed at the right of the numeral to act as place holders and fill empty spaces in the column, as in Examples 1 and 2.

EXAMPLES

1. $0.858 + 0.7 = ?$
$0.7 = 0.700$

Recopying:
$$\begin{array}{r} 0.858 \\ \underline{0.700} \\ 1.558 \end{array}$$

2. $\$16.56 + \$5 = ?$
We know that $\$5 = \$5.00$ Why?
Then,
$$\begin{array}{r} \$16.56 \\ \underline{5.00} \\ \$21.56 \end{array}$$

In the Example at the right, you cannot annex zeros to 0.7 lb. because the measurement was to the nearest tenth of a pound, and you do not know what digits may have been dropped. Instead, round all other numbers to the same number of decimal places as the one with the least number of places.

EXAMPLE

0.858 lb. + 0.7 lb. = ?
Round 0.858 to 0.9. Why?
Then,
$$\begin{array}{r} 0.9 \\ \underline{0.7} \\ 1.6 \end{array}$$
Answer: 1.6 lb.

Copy and add each of the following: Round when necessary.

1. $16.251 + 0.3 + 27.008 + 714.6$
2. $78 + 15.3 + 1.007 + 8.59$
3. $\$129.07 + \$6 + \$77.30 + \$.01$
4. 39.2 ft. + 62.35 ft. + 16.005 ft.
5. $\$50 + \$175.76 + \$.91 + \$17$
6. 0.07 in. + 1.31 in. + 0.8 in.
7. $0.4 + 0.038 + 1.85 + 0.0916$
8. $25.01 + 37.984 + 15.07 + 18.3$
9. $0.103 + 2.58 + 75.929 + 3.807$
10. $\$125 + \$378.40 + \$126.48 + \$.28 + \$7.52$
11. 0.3715 lb. + 1.43 lb. + 185.917 lb. + 3.689 lb. + 26.348 lb.
12. $\$5 + \$2.75 + \$18 + \$.75$
13. 0.15 in. + 2.1 in. + 6.75 in. + 5.0 in.
14. $0.039 + 17.5 + 9.25 + 0.675$

B. *Addition and decimals*

The table at the right shows the names of the places to the right of the decimal point. Examine it, for it will help you in reading and writing numerals for small numbers. The number named in the table is read: "Three hundred sixty nine thousand sixty five millionths."

tenths	hundredths	thousandths	ten-thousandths	hundred-thousandths	millionths
.3	6	9	0	6	5

At the right three numbers are named by decimals. The first, a, is read five thousandths. How many places does it have?

a. 0.005
b. 0.0009
c. 0.004916

b is read nine ten-thousandths. How many places does it have?
c is read four thousand nine hundred sixteen millionths.

Write numerals to represent the following:

1. One thousand four hundred and six hundredths
2. Four hundred nine thousandths
3. Sixty and four tenths
4. Twenty eight thousand thirty six
5. Four million, seven hundred sixty five thousand, seven hundred nine and five tenths
6. Twelve hundredths
7. Two billion, seven hundred six thousand, two hundred seventy eight
8. Eighteen ten-thousandths

Write each of the following as decimals. Add. Remember to align the decimal points one above the other.

9. 3 tenths + 5 tenths
10. 9 tenths + 6 tenths
11. 3 hundredths + 18 hundredths
12. 29 hundredths + 18 hundredths
13. 43 ten-thousandths + 187 thousandths
14. 206 ten-thousandths + 47 thousandths + 7 thousandths
15. 42 thousandths + 159 ten-thousandths + 9 thousandths
16. 18 thousandths + 305 thousandths + 6 hundredths
17. 9 hundredths + 285 thousandths + 5 tenths
18. 305 thousandths + 17 ten thousandths + 13 hundredths

C. *Keeping the decimal points under each other*

When subtracting using decimals, keep the decimal points under each other in a column. To do this you must:

a. Keep tenths under tenths, hundredths under hundredths, and so on.
b. If the number of decimal places in the two numerals are unequal, they should be made equal before subtracting.
c. If the numerals represent money or something else other than measurement, annex zeros as necessary.
d. If the numerals are from measurements, round to the same number of decimal places as the one having the least number of places.

EXAMPLES

1. 8 lb. − 3.25 lb. = ?
Round 3.25 lb. to 3 lb.

$$\begin{array}{r} 8 \text{ lb.} \\ -\ 3 \text{ lb.} \\ \hline 5 \text{ lb.} \end{array}$$

2. \$5 − \$3.15 = ?

$$\begin{array}{r} \$5.00 \\ -\ 3.15 \\ \hline \$1.85 \end{array}$$

3. 15.23 − 8.4 = ?

$$\begin{array}{r} 15.23 \\ -\ 8.40 \\ \hline 6.83 \end{array}$$

Subtract:

1. $\begin{array}{r} 7.94 \\ -\ 6.32 \\ \hline \end{array}$

2. $\begin{array}{r} \$8 \\ -\ 5.88 \\ \hline \end{array}$

3. $\begin{array}{r} 38.45 \\ -\ 16.8 \\ \hline \end{array}$

4. $\begin{array}{r} 27.47 \\ -\ 16.89 \\ \hline \end{array}$

5. $\begin{array}{r} 8.35 \text{ lb.} \\ -\ 6.21 \text{ lb.} \\ \hline \end{array}$

6. $\begin{array}{r} 9.17 \text{ ft.} \\ -\ 1.88 \text{ ft.} \\ \hline \end{array}$

7. $\begin{array}{r} 15.63 \text{ in.} \\ -\ 12.47 \text{ in.} \\ \hline \end{array}$

8. $\begin{array}{r} 37.25 \text{ mi.} \\ -\ 6.49 \text{ mi.} \\ \hline \end{array}$

9. $\begin{array}{r} 35.625 \\ -\ 16.732 \\ \hline \end{array}$

10. $\begin{array}{r} 125.436 \\ -\ 65.982 \\ \hline \end{array}$

11. $\begin{array}{r} \$5 \\ -\ 3.30 \\ \hline \end{array}$

12. $\begin{array}{r} \$9 \\ -\ 5.73 \\ \hline \end{array}$

13. $\begin{array}{r} \$15 \\ -\ 12.30 \\ \hline \end{array}$

14. $\begin{array}{r} \$28 \\ -\ 18.74 \\ \hline \end{array}$

15. $\begin{array}{r} 175 \text{ ft.} \\ -\ 135.642 \text{ ft.} \\ \hline \end{array}$

16. $\begin{array}{r} \$356 \\ -\ 74.25 \\ \hline \end{array}$

17. $2568.43
− 1752.59

18. $1541.86
− 89.37

19. $1305.17
−1105.18

20. $1560
− 970.05

21. $200.00
− 1.75

22. $811.75
− 59.18

23. $905.05
− 18.50

24. $834.77
− 345.78

25. $1300.05
− 1101.16

26. $206.73
− 180.00

27. $403.20
−85.17

28. $700
− 319.06

D. *Subtraction and decimals*

Subtract: Remember to keep the decimal points one under the other.

1. 17 hundredths from 5 tenths
2. 209 thousandths from 57 hundredths
3. 3 thousandths from 715 ten-thousandths
4. 852 ten-thousandths from 9 tenths
5. 75 thousandths from 25 hundredths
6. 839 thousandths from 99 hundredths
7. 4 hundredths from 5 tenths
8. 25 thousandths from 715 ten-thousandths
9. 761 ten-thousandths from 8 hundredths
10. 8 hundredths from 1 tenth
11. 206 ten-thousandths from 1 tenth
12. 9 tenths from 975 thousandths
13. 47 hundredths from 756 thousandths
14. 1 ten-thousandth from 1 thousandth
15. 18 thousandths from 5 hundredths
16. 1568 ten-thousandths from 74 hundredths
17. 56 hundredths from 732 thousandths
18. 3 tenths from 3565 ten-thousandths
19. 25 thousandths from 8 tenths
20. 309 ten-thousandths from 43 thousandths

PRACTICE TEST

A. *Add:* Round when necessary.

1. 1.5 in. + 2.67 in. + 3.906 in. + 159.3967 in.
2. 99.7 + 8.29 + 25.009 + 3.711
3. 9.003 lb. + 1.13 lb. + 2.63 lb. + 0.0408 lb.
4. 0.0601 + 2.31 + 100.3 + 9.8

B. Write as numerals. Add.

1. 3 and 4 ten-thousandths + 12 thousandths + 8 tenths
2. 8 hundredths + 5 and 1 thousandth + 25 ten-thousandths
3. 103 ten-thousandths + 7 thousandths + 16 hundredths
4. 25 thousandths + 5 hundredths + 1 tenth

C. *Subtract:* Round when necessary.

1. 1.896 − 0.9
2. 9.07 − 0.004
3. 25.506 − 10.9
4. 36.09 ft. − 21.6 ft.
5. 8.103 in. − 7.86 in.
6. 10.3 lb. − 9 lb.

D. Write as numerals. Subtract.

1. 5 hundredths from 2 and 1 tenth
2. 26 thousandths from 18 hundredths
3. 48 ten-thousandths from 13 thousandths
4. 301 ten-thousandths from 5 and 1 tenth

PRACTICE EXERCISES

A. *Locating the decimal point in the product*

Rule: Add the number of places to the right of the decimal point in each of the factors. This sum represents the number of places to the right of the decimal point in the product.

Explain how the decimal point was located in each of these examples.

1. $4.2 \times 0.3 = 1.26$
2. $14.7 \times 0.13 = 1.911$
3. $0.002 \times 15.4 = 0.0308$
4. $0.07 \times 0.004 = 0.00028$
5. $15 \times 1.5 = 22.5$
6. $1.4 \times 0.49 = 0.686$
7. $2.5 \times 7.5 = 18.75$
8. $0.008 \times 0.9 = 0.0072$

Copy the product and correctly place the decimal point. Annex zeros when needed.

9. $224 \times 2.4 = 5376$
10. $22.4 \times 0.24 = 5376$
11. $2.24 \times 0.024 = 5376$
12. $0.0224 \times 0.24 = 5376$
13. $0.688 \times 0.45 = 30960$
14. $68.8 \times 0.45 = 30960$
15. $6.88 \times 0.45 = 30960$
16. $6.88 \times 4.5 = 30960$
17. $3.61 \times 5.7 = 20577$
18. $0.361 \times 5.7 = 20577$
19. $36.1 \times 5.7 = 20577$
20. $0.361 \times 0.057 = 20577$

B. *Finding the product; decimals*

In multiplication with decimals, we first find the product as in multiplying whole numbers. Then we locate the decimal point according to the Rule in A on the previous page.

In Example 1 below, there are _?_ decimal place(s) in 3.93 and _?_ decimal place(s) in 41.3. Therefore, there are _?_ decimal place(s) in the product.

EXAMPLES

1.
```
    3.93
    41.3
    ----
    1179
    393
  1572
 -------
 162.309
```

2.
```
   3.88
   0.02
 ------
 0.0776
```

We must have how many decimal places in the product in Example 2 above? Why did we put a zero to the left of 7 in the product?

Find the products doing as little written work as possible.

1. 203×0.03
2. 1.3×0.4
3. 0.21×0.04
4. 49×0.002
5. 1.4×0.07
6. 20×0.32
7. 1.6×0.4
8. 0.14×0.7
9. 2.5×0.3
10. 3.2×3
11. 0.62×20
12. 41×0.03
13. 0.17×5
14. 0.31×0.003
15. 4.1×0.02
16. 20.5×0.003
17. 0.13×0.4
18. 1.2×0.07
19. 2.3×0.003
20. 4.1×0.07

Find the products to the nearest cent.

21. $3.56 \times \$256$

22. $41{,}450 \times \$4.93$

23. $61{,}684 \times \$.075$

24. $71{,}850 \times \$.066$

25. $16.88 \times \$.1562$

26. $287.8 \times \$.3158$

27. $23.680 \times \$5.07$

28. $81.348 \times \$.247$

29. $42{,}552 \times \$.508$

30. $985.8 \times \$.093$

31. $4.892 \times \$.139$

32. $3.657 \times \$.099$

C. *Locating the decimal point by estimation*

To find the location of the decimal point by estimation, first determine the size of the quotient in powers of 10. Referring to the example, answer these questions. Is the quotient:

EXAMPLE

$$\begin{array}{r} 3870 \\ 0.0216_{\wedge}\overline{)83.5920_{\wedge}} \end{array}$$

a. greater than 10?

b. greater than 100?

c. greater than 1000?

d. greater than 10,000?

If you answered correctly, you found that the quotient was between 1000 and 10,000; that is

$$1000 < q < 10{,}000$$

Use this method for locating the decimal point in each of the following exercises. Copy the quotient and locate the decimal point in the proper place. Annex zeros as necessary.

1. $\begin{array}{r} 89 \\ 4\overline{)3.56} \end{array}$

2. $\begin{array}{r} 412 \\ 0.02\overline{)82.4} \end{array}$

3. $\begin{array}{r} 141 \\ 0.7\overline{)9.87} \end{array}$

4. $\begin{array}{r} 59 \\ 5\overline{)0.295} \end{array}$

5. $\begin{array}{r} 8 \\ 0.03\overline{)24.} \end{array}$

6. $\begin{array}{r} 84 \\ 0.8\overline{)67.2} \end{array}$

7. $\begin{array}{r} 93 \\ 0.09\overline{)8.37} \end{array}$

8. $\begin{array}{r} 627 \\ 0.01\overline{)6.27} \end{array}$

9. $26.88 \div 2.4 = 112$

10. $65.2795 \div 7.15 = 913$

11. $\begin{array}{r} 64 \\ 0.4\overline{)2.56} \end{array}$

12. $\begin{array}{r} 82 \\ 0.06\overline{)4.92} \end{array}$

13. $\begin{array}{r} 32 \\ 9\overline{)2.88} \end{array}$

14. $\begin{array}{r} 172 \\ 0.02\overline{)3.44} \end{array}$

15. $\begin{array}{r} 45 \\ 1.5\overline{)67.5} \end{array}$

16. $\begin{array}{r} 98 \\ 0.07\overline{)6.86} \end{array}$

17. $\begin{array}{r} 27 \\ 9.5\overline{)2.565} \end{array}$

D. *Division and decimals*

When the divisor has a decimal point, the first thing to do is to multiply both divisor and dividend by 10, 100, 1000 or a power of 10 that will make the divisor a whole number. In the Example, both divisor and dividend were multiplied by what number?

How is the new location of the decimal point in divisor and dividend indicated?

How was the decimal point in the quotient located? Check the location by estimation.

EXAMPLE

Divide 1.05 by 12.5

$$\begin{array}{r} .084 \\ 12.5_{\wedge}\overline{)1.0_{\wedge}500} \\ \underline{1\,0\,00} \\ 500 \\ \underline{500} \end{array}$$

$$? < q < ?$$

Find the quotients. Check the location of the decimal point by estimation.

1. $0.8\overline{)0.96}$

2. $0.9\overline{)0.63}$

3. $0.8\overline{)37.6}$

4. $0.5\overline{)0.05}$

5. $0.7\overline{)0.056}$

6. $0.06\overline{)0.036}$

7. $0.06\overline{)97.2}$

8. $0.05\overline{)0.735}$

9. $0.008\overline{)496}$

10. $0.01\overline{)67.2}$

11. $0.917\overline{)2.34752}$

12. $0.726\overline{)522.72}$

13. $1.5\overline{)7.65}$

14. $0.18\overline{)648}$

15. $0.006\overline{)432}$

16. $0.14\overline{)39.2}$

17. $1.7\overline{)49.13}$

18. $0.24\overline{)11.52}$

19. $1.9\overline{)0.722}$

20. $23\overline{)12.167}$

21. $0.013\overline{)4.394}$

22. $2.2\overline{)9.68}$

Find the quotient to the nearest cent.

23. \$16.38 ÷ 0.095

24. \$56.73 ÷ 4.87

25. \$27.61 ÷ 0.247

26. \$13.32 ÷ 0.402

27. \$3685 ÷ 0.025

28. \$1656 ÷ 5.07

29. \$56.80 ÷ 0.167

30. \$38.12 ÷ 0.353

A. Copy each product, and correctly place the decimal point.

1. $0.16 \times 0.3 = 48$
2. $3.2 \times 13 = 416$
3. $2.09 \times 1.2 = 2508$
4. $3.19 \times 2.6 = 8294$
5. $12.85 \times 0.015 = 19275$
6. $70.5 \times 2.94 = 207270$
7. $0.14 \times 1.96 = 2744$
8. $1.9 \times 3.61 = 6859$

B. Find the product of each of the following:

1. 6.8×8.6
2. 0.687×4.5
3. 5.37×0.76
4. 5.11×6.08
5. 32.9×18.2
6. 0.05×0.009
7. 0.15×3.4
8. 0.043×0.19

C. Find the products without any written computation.

1. 5.07×10
2. 13.29×1000
3. $5.625 \times 10{,}000$
4. 0.5832×0.01
5. 0.007×0.001
6. 1.32×200
7. 14.3×30
8. 0.115×400
9. 3.122×0.02
10. 1.026×0.03

D. Copy each quotient, and correctly place the decimal point.

1. $24.6 \div 6 = 41$
2. $8.48 \div 40 = 212$
3. $5 \div 25 = 2$
4. $3.64 \div 1.4 = 26$
5. $3.366 \div 0.6 = 561$
6. $6 \div 0.75 = 8$
7. $4.59 \div 1.5 = 306$
8. $12 \div 0.025 = 48$

E. Find the quotient of each of the following:

1. $4.242 \div 7$
2. $\$23.94 \div 6.3$
3. $56.48 \div 0.16$
4. $0.0378 \div 1.4$
5. $\$256 \div 3.2$
6. $54 \div 0.025$
7. $2.88 \div 0.012$
8. $0.768 \div 1.6$

F. Find the quotients to the nearest cent.

1. $\$20.90 \div 0.181$
2. $\$10.92 \div 0.2553$
3. $\$600.62 \div 0.257$
4. $\$56.17 \div 0.505$
5. $\$98.13 \div 4.86$
6. $\$1548.89 \div 0.069$
7. $\$41.58 \div 1.8$
8. $\$78.39 \div 0.27$

A. *Add:* Write all answers in simplest form.

1. $\begin{array}{r} 7\frac{1}{3} \\ 5\frac{3}{5} \\ 10\frac{1}{15} \\ 2\frac{2}{3} \\ \hline \end{array}$

2. $\begin{array}{r} 3\frac{3}{16} \\ 4\frac{1}{4} \\ 12\frac{1}{8} \\ 24\frac{5}{24} \\ \hline \end{array}$

3. $\begin{array}{r} 9\frac{1}{2} \\ 4\frac{1}{5} \\ 6\frac{3}{8} \\ 14\frac{7}{10} \\ \hline \end{array}$

4. $\begin{array}{r} 13\frac{3}{7} \\ 8\frac{1}{4} \\ 9\frac{5}{14} \\ 27\frac{1}{2} \\ \hline \end{array}$

5. $\begin{array}{r} 22\frac{5}{6} \\ 8\frac{2}{3} \\ 10\frac{3}{10} \\ 4\frac{1}{5} \\ \hline \end{array}$

6. $\begin{array}{r} 7\frac{5}{9} \\ 11\frac{1}{3} \\ 4\frac{4}{15} \\ 3\frac{1}{5} \\ \hline \end{array}$

7. $\begin{array}{r} 4\frac{7}{12} \\ 9\frac{1}{2} \\ 17\frac{3}{8} \\ 4\frac{2}{3} \\ \hline \end{array}$

8. $\begin{array}{r} 23\frac{2}{9} \\ 6\frac{5}{12} \\ 10\frac{1}{4} \\ 3\frac{7}{18} \\ \hline \end{array}$

9. $\begin{array}{r} 4\frac{5}{16} \\ 17\frac{5}{9} \\ 3\frac{2}{3} \\ 14\frac{1}{2} \\ \hline \end{array}$

B. *Subtract:* Write all answers in simplest form.

1. $\begin{array}{r} 35\frac{3}{10} \\ 24\frac{2}{15} \\ \hline \end{array}$

2. $\begin{array}{r} 15\frac{7}{8} \\ 12\frac{1}{3} \\ \hline \end{array}$

3. $\begin{array}{r} 46\frac{2}{5} \\ 22\frac{3}{4} \\ \hline \end{array}$

4. $\begin{array}{r} 75\frac{2}{3} \\ 51\frac{3}{8} \\ \hline \end{array}$

5. $\begin{array}{r} 54\frac{7}{16} \\ 32\frac{7}{8} \\ \hline \end{array}$

6. $\begin{array}{r} 24\frac{2}{3} \\ 11\frac{5}{6} \\ \hline \end{array}$

7. $\begin{array}{r} 55\frac{5}{6} \\ 42\frac{2}{3} \\ \hline \end{array}$

8. $\begin{array}{r} 67\frac{1}{10} \\ 53\frac{1}{5} \\ \hline \end{array}$

9. $\begin{array}{r} 38\frac{1}{4} \\ 23\frac{2}{3} \\ \hline \end{array}$

10. $\begin{array}{r} 96\frac{1}{3} \\ 72\frac{3}{8} \\ \hline \end{array}$

11. $\begin{array}{r} 31\frac{1}{15} \\ 22\frac{3}{5} \\ \hline \end{array}$

12. $\begin{array}{r} 26\frac{7}{12} \\ 18\frac{7}{9} \\ \hline \end{array}$

13. $\begin{array}{r} 45\frac{3}{8} \\ 29\frac{5}{12} \\ \hline \end{array}$

14. $\begin{array}{r} 68\frac{3}{7} \\ 43\frac{1}{4} \\ \hline \end{array}$

15. $\begin{array}{r} 17 \\ 14\frac{6}{7} \\ \hline \end{array}$

C. *Multiply:* Write all answers in simplest form.

1. $15 \times \frac{3}{5}$
2. $\frac{5}{8} \times 24$
3. $\frac{15}{16} \times 80$
4. $9 \times \frac{5}{6}$
5. $18 \times \frac{5}{9}$
6. $16 \times \frac{7}{8}$
7. $\frac{5}{8} \times \frac{4}{5}$
8. $\frac{3}{16} \times \frac{4}{9}$
9. $\frac{4}{5} \times \frac{5}{7}$
10. $\frac{2}{3} \times \frac{7}{8}$
11. $1\frac{4}{7} \times \frac{5}{8}$
12. $2\frac{8}{9} \times \frac{3}{7}$
13. $1\frac{2}{5} \times \frac{5}{8}$
14. $\frac{15}{16} \times 2\frac{4}{5}$
15. $6\frac{2}{3} \times \frac{4}{5}$
16. $32 \times 6\frac{1}{4}$
17. $36 \times 3\frac{1}{3}$
18. $24 \times 8\frac{1}{3}$
19. $4\frac{1}{2} \times \frac{2}{3}$
20. $\frac{5}{7} \times 3\frac{1}{2}$
21. $12\frac{1}{2} \times 3\frac{3}{5}$
22. $14\frac{2}{7} \times 4\frac{9}{10}$
23. $18\frac{1}{3} \times \frac{9}{11}$
24. $17\frac{1}{2} \times 6\frac{2}{7}$
25. $13\frac{1}{3} \times 7\frac{1}{5}$
26. $20\frac{1}{6} \times 8\frac{2}{11}$
27. $62\frac{1}{2} \times 4\frac{2}{25}$

D. *Divide:* Write all answers in simplest form.

1. $\frac{3}{5} \div 24$
2. $132 \div \frac{11}{12}$
3. $\frac{3}{4} \div 75$
4. $\frac{5}{12} \div 15$
5. $16 \div \frac{2}{3}$
6. $5\frac{3}{5} \div 4$
7. $3\frac{1}{3} \div 20$
8. $5\frac{1}{3} \div 6$
9. $4\frac{1}{8} \div 11$
10. $2\frac{2}{7} \div 8$
11. $\frac{3}{8} \div \frac{2}{3}$
12. $\frac{7}{8} \div \frac{15}{16}$
13. $\frac{5}{9} \div \frac{2}{3}$
14. $\frac{15}{16} \div \frac{3}{8}$
15. $\frac{4}{5} \div \frac{8}{15}$
16. $15\frac{5}{9} \div \frac{5}{18}$
17. $3\frac{3}{5} \div \frac{6}{25}$
18. $18\frac{2}{5} \div \frac{5}{6}$
19. $2\frac{1}{7} \div \frac{5}{8}$
20. $2\frac{1}{3} \div \frac{7}{8}$
21. $\frac{3}{8} \div \frac{15}{16}$

PRACTICE TEST

A. *Add:* Write all answers in simplest form.

1. $\begin{array}{r} 6\frac{1}{12} \\ 5\frac{1}{6} \\ 19\frac{2}{3} \\ 4\frac{1}{2} \\ \hline \end{array}$

2. $\begin{array}{r} 8\frac{1}{2} \\ 6\frac{3}{4} \\ 24\frac{5}{8} \\ 17\frac{5}{6} \\ \hline \end{array}$

3. $\begin{array}{r} 26\frac{3}{5} \\ 9\frac{8}{9} \\ 41\frac{2}{3} \\ 12\frac{7}{15} \\ \hline \end{array}$

4. $\begin{array}{r} 8\frac{3}{5} \\ 13\frac{3}{4} \\ 12\frac{1}{8} \\ 3\frac{7}{10} \\ \hline \end{array}$

5. $\begin{array}{r} 9\frac{2}{3} \\ 13\frac{3}{4} \\ 8\frac{2}{9} \\ 12\frac{11}{12} \\ \hline \end{array}$

6. $\begin{array}{r} 12\frac{7}{10} \\ 19\frac{3}{4} \\ 13\frac{3}{5} \\ 6\frac{1}{2} \\ \hline \end{array}$

B. *Subtract:* Write all answers in simplest form.

1. $\begin{array}{r} 29\frac{3}{10} \\ 20\frac{3}{5} \\ \hline \end{array}$

2. $\begin{array}{r} 31\frac{3}{4} \\ 25\frac{5}{6} \\ \hline \end{array}$

3. $\begin{array}{r} 37\frac{5}{12} \\ 19\frac{3}{4} \\ \hline \end{array}$

4. $\begin{array}{r} 19\frac{3}{5} \\ 11\frac{1}{8} \\ \hline \end{array}$

5. $\begin{array}{r} 33\frac{1}{2} \\ 29\frac{3}{5} \\ \hline \end{array}$

6. $\begin{array}{r} 15\frac{1}{4} \\ 10\frac{5}{6} \\ \hline \end{array}$

C. *Multiply:* Write all answers in simplest form.

1. $18 \times \frac{2}{9}$
2. $16 \times \frac{3}{8}$
3. $6\frac{2}{3} \times \frac{3}{5}$
4. $36 \times 6\frac{1}{4}$
5. $\frac{5}{8} \times \frac{11}{12}$
6. $\frac{9}{14} \times \frac{7}{18}$
7. $\frac{4}{7} \times \frac{5}{8}$
8. $\frac{8}{9} \times \frac{3}{7}$
9. $4\frac{2}{5} \times 4\frac{1}{11}$
10. $7\frac{1}{2} \times 3\frac{1}{3}$
11. $6\frac{2}{3} \times 9\frac{3}{25}$
12. $8\frac{4}{5} \times 4\frac{1}{11}$

D. *Divide:* Write all answers in simplest form.

1. $24\frac{1}{5} \div \frac{11}{12}$
2. $18\frac{2}{3} \div \frac{7}{8}$
3. $37\frac{1}{2} \div \frac{15}{16}$
4. $13\frac{1}{3} \div \frac{8}{9}$
5. $6\frac{2}{3} \div \frac{5}{6}$
6. $3 \div 1\frac{1}{8}$
7. $20 \div \frac{10}{11}$
8. $1\frac{7}{9} \div 5\frac{1}{3}$
9. $1\frac{3}{5} \div 4\frac{4}{5}$
10. $16\frac{1}{2} \div 6\frac{2}{3}$
11. $1\frac{5}{8} \div 6\frac{1}{2}$
12. $16\frac{2}{3} \div 14\frac{2}{7}$

A. Copy each product and insert the decimal point in the proper place in the product. Annex zeros when necessary.

Rule: Add the number of places to the right of the decimal point in each of the factors. This sum represents the number of places to the right of the decimal point in the product.

1. $3.35 \times 4.5 = 15075$

2. $0.0335 \times 4.5 = 15075$

3. $335 \times 0.45 = 15075$

4. $0.335 \times 4.5 = 15075$

5. $0.756 \times 0.024 = 18144$

6. $0.756 \times 2.4 = 18144$

7. $756 \times 0.24 = 18144$

8. $0.756 \times 0.24 = 18144$

9. $9.13 \times 7.15 = 652795$

10. $0.913 \times 0.715 = 652795$

11. $91.3 \times 7.15 = 652795$

12. $0.913 \times 0.00715 = 652795$

B. Find the product of each of the following:

Rule: Multiply as you do with whole numbers, and place the decimal point as per the Rule stated in A above.

1. $\begin{array}{r} 248 \\ \underline{0.25} \end{array}$

2. $\begin{array}{r} 5.06 \\ \underline{0.125} \end{array}$

3. $\begin{array}{r} 60.7 \\ \underline{0.231} \end{array}$

4. $\begin{array}{r} 80.2 \\ \underline{0.105} \end{array}$

5. $\begin{array}{r} 412 \\ \underline{0.35} \end{array}$

6. $\begin{array}{r} 0.306 \\ \underline{16.2} \end{array}$

7. $\begin{array}{r} 0.362 \\ \underline{4.15} \end{array}$

8. $\begin{array}{r} 401.4 \\ \underline{32.03} \end{array}$

9. $\begin{array}{r} 6.01 \\ \underline{0.14} \end{array}$

10. $\begin{array}{r} 0.402 \\ \underline{0.54} \end{array}$

11. $\begin{array}{r} 20.4 \\ \underline{3.14} \end{array}$

12. $\begin{array}{r} 30.12 \\ \underline{6.042} \end{array}$

13. $\begin{array}{r} 55.3 \\ \underline{0.004} \end{array}$

14. $\begin{array}{r} 6.25 \\ \underline{11.2} \end{array}$

15. $\begin{array}{r} 20.4 \\ \underline{0.072} \end{array}$

16. $\begin{array}{r} 3.025 \\ \underline{17.1} \end{array}$

C. Insert the decimal point in each quotient. Annex zeros when necessary. To locate the decimal point, estimate the size of the quotient. Is the quotient:

a. greater than 10? b. greater than 100? c. greater than 1000?

1. $10.320 \div 0.15 = 688$
2. $411.54 \div 5.7 = 722$
3. $15.075 \div 45 = 335$
4. $1.8144 \div 2.4 = 756$
5. $107.52 \div 0.112 = 96$
6. $7.2240 \div 0.688 = 105$
7. $12.3462 \div 72.2 = 171$
8. $10.5525 \div 0.335 = 315$

D. Find the quotients.

Rule: Multiply the divisor and dividend by a power of 10 (10, 100, 1000, etc.) in order to make the divisor a whole number, then divide.

---

1. $0.4\overline{)4.032}$
2. $0.7\overline{)28.14}$
3. $0.7\overline{)42}$
4. $0.08\overline{)168}$
5. $0.004\overline{)376}$
6. $1.2\overline{)2460}$
7. $0.04\overline{)49.28}$
8. $0.006\overline{)4.44}$
9. $0.8\overline{)0.928}$
10. $0.009\overline{)5.94}$
11. $0.21\overline{)44.1}$
12. $0.2\overline{)356}$

PRACTICE TEST

A. Copy each product and correctly place the decimal point.

1. $14.32 \times 2.1 = 30072$
2. $0.215 \times 0.12 = 2580$
3. $0.031 \times 0.0023 = 713$
4. $2.53 \times 3.1 = 7843$
5. $23.9 \times 3.05 = 72895$
6. $1.25 \times 0.005 = 625$

B. Find the product of each of the following:

1. 19.03×2.04
2. 45.6×2.005
3. 9.8×0.0024
4. 16.246×0.12
5. 3.75×1.13
6. 0.0056×0.00025

C. Copy each quotient and correctly place the decimal point.

1. $38.8212 \div 2.04 = 1903$
2. $91.4280 \div 45.6 = 2005$
3. $0.02352 \div 0.0024 = 98$
4. $4.2375 \div 1.13 = 375$
5. $1.94952 \div 16.246 = 12$
6. $2.4 \div 0.006 = 4$

D. Find the quotient of each of the following:

1. $1.5\overline{)225}$
2. $0.07\overline{)4.2}$
3. $0.04\overline{)0.0064}$
4. $0.012\overline{)0.0036}$
5. $0.025\overline{)3.25}$
6. $0.0025\overline{)325}$

PRACTICE EXERCISES

A. *Addition*

To add fractional numbers, first find the least common multiple (L.C.M.) of the denominators. The L.C.M. of 2, 5, and 4 is 20. Note that 40, 80, 160, etc., are also common multiples of 2, 5, and 4. However, 20 is the *least* common multiple. If the denominators do not have a common factor, such as the denominators 2, 5, and 7, the L.C.M. is simply found by multiplying the denominators together. Thus, the L.C.M. of 2, 5, and 7 is $2 \times 5 \times 7 = 70$.

1. $2\frac{1}{3} + 1\frac{3}{4}$
2. $7\frac{1}{3} + 5\frac{2}{5}$
3. $3\frac{2}{9} + 1\frac{1}{3}$
4. $4\frac{1}{5} + 3\frac{6}{7}$
5. $9\frac{5}{6} + 2\frac{3}{4}$
6. $4\frac{5}{8} + \frac{4}{5}$
7. $6\frac{1}{2} + 2\frac{3}{7}$
8. $1\frac{8}{9} + \frac{7}{10}$
9. $1\frac{2}{5} + 4\frac{1}{2}$

10. $\begin{array}{r} 2\frac{1}{3} \\ 4\frac{1}{4} \\ \underline{7\frac{5}{6}} \end{array}$

11. $\begin{array}{r} 3\frac{1}{2} \\ 2\frac{2}{5} \\ \underline{5\frac{1}{4}} \end{array}$

12. $\begin{array}{r} 7 \\ 6\frac{3}{8} \\ \underline{3\frac{3}{4}} \end{array}$

13. $\begin{array}{r} 3\frac{1}{8} \\ 2\frac{1}{16} \\ \underline{1\frac{1}{2}} \end{array}$

14. $\begin{array}{r} 4\frac{1}{3} \\ 3\frac{1}{2} \\ \underline{2} \end{array}$

15. $\begin{array}{r} 12\frac{7}{10} \\ 19\frac{3}{4} \\ \underline{13\frac{3}{5}} \end{array}$

16. $\begin{array}{r} 6\frac{2}{3} \\ 4\frac{5}{6} \\ \underline{2\frac{1}{8}} \end{array}$

17. $\begin{array}{r} 3\frac{8}{15} \\ 7\frac{1}{5} \\ \underline{3\frac{7}{10}} \end{array}$

18. $\begin{array}{r} 4\frac{2}{5} \\ 1\frac{1}{3} \\ \underline{2\frac{1}{7}} \end{array}$

19. $\begin{array}{r} 15\frac{3}{16} \\ 4\frac{5}{8} \\ \underline{1\frac{1}{4}} \end{array}$

20. $\begin{array}{r} 14\frac{3}{4} \\ 16\frac{1}{2} \\ \underline{9\frac{3}{8}} \end{array}$

21. $\begin{array}{r} 17\frac{1}{3} \\ 5\frac{1}{6} \\ \underline{4\frac{1}{2}} \end{array}$

22. $\begin{array}{r} 19\frac{3}{5} \\ 6\frac{7}{10} \\ \underline{12\frac{4}{15}} \end{array}$

23. $\begin{array}{r} 15\frac{4}{5} \\ 17\frac{1}{2} \\ \underline{3\frac{3}{10}} \end{array}$

24. $\begin{array}{r} 8\frac{3}{8} \\ 12\frac{3}{5} \\ \underline{5\frac{7}{12}} \end{array}$

25. $\begin{array}{r} 16\frac{1}{2} \\ 18\frac{2}{3} \\ \underline{3\frac{5}{6}} \end{array}$

26. $\begin{array}{r} 12\frac{7}{10} \\ 19\frac{3}{4} \\ \underline{13\frac{3}{5}} \end{array}$

27. $\begin{array}{r} 4\frac{5}{16} \\ 17\frac{5}{9} \\ \underline{10\frac{1}{3}} \end{array}$

B. *Subtraction*

Study the Example at the right.

EXAMPLE

$$\begin{array}{r} 5\frac{1}{3} = 5\frac{5}{15} = 4\frac{20}{15} \\ \underline{4\frac{2}{5}} = \underline{4\frac{6}{15}} = \underline{4\frac{6}{15}} \\ \frac{14}{15} \end{array}$$

1. $\begin{array}{r} 2\frac{11}{12} \\ \underline{1\frac{5}{18}} \end{array}$	**4.** $\begin{array}{r} 3\frac{7}{9} \\ \underline{2\frac{3}{4}} \end{array}$	**7.** $\begin{array}{r} 4\frac{3}{4} \\ \underline{2\frac{1}{6}} \end{array}$
2. $\begin{array}{r} 3\frac{9}{16} \\ \underline{3\frac{1}{3}} \end{array}$	**5.** $\begin{array}{r} 8\frac{1}{5} \\ \underline{5\frac{5}{6}} \end{array}$	**8.** $\begin{array}{r} 4\frac{1}{8} \\ \underline{1\frac{1}{6}} \end{array}$
3. $\begin{array}{r} 5\frac{1}{2} \\ \underline{3\frac{3}{8}} \end{array}$	**6.** $\begin{array}{r} 18\frac{2}{3} \\ \underline{15\frac{1}{2}} \end{array}$	**9.** $\begin{array}{r} 7\frac{5}{8} \\ \underline{6\frac{7}{12}} \end{array}$

C. *Multiplication*

Study the Example and complete these statements.

Expressed as improper fractions,

$16\frac{1}{4} = ?$ $\qquad 5\frac{3}{15} = ?$

65 and 15 are both divisible by __?__.
4 and 78 are both divisible by __?__.
3 and 39 are both divisible by __?__.
In simplest form the product is __?__.

EXAMPLE

$16\frac{1}{4} \times 5\frac{3}{15}$

$$\frac{\overset{13}{\cancel{65}}}{\underset{2}{\cancel{4}}} \times \frac{\overset{\overset{13}{39}}{\cancel{78}}}{\underset{\underset{1}{3}}{\cancel{15}}} = ?$$

Multiply: Write all answers in simplest form.

1. $\frac{3}{8} \times \frac{5}{12}$	**13.** $16\frac{2}{3} \times \frac{4}{15}$	**25.** $13\frac{1}{3} \times 1\frac{7}{8}$
2. $\frac{3}{14} \times \frac{7}{8}$	**14.** $18\frac{3}{5} \times \frac{5}{31}$	**26.** $2\frac{7}{10} \times 8\frac{8}{9}$
3. $\frac{4}{5} \times \frac{1}{2}$	**15.** $2\frac{4}{5} \times 6\frac{1}{4}$	**27.** $6\frac{3}{5} \times 3\frac{2}{11}$
4. $\frac{7}{8} \times \frac{4}{21}$	**16.** $\frac{8}{9} \times 7\frac{7}{8}$	**28.** $8\frac{1}{3} \times 5\frac{2}{5}$
5. $\frac{4}{15} \times \frac{5}{8}$	**17.** $2\frac{7}{16} \times 13\frac{2}{3}$	**29.** $17\frac{1}{4} \times 3\frac{1}{3}$
6. $\frac{5}{8} \times \frac{2}{3}$	**18.** $2\frac{5}{8} \times 9\frac{1}{7}$	**30.** $16\frac{2}{3} \times 8\frac{2}{5}$
7. $\frac{3}{8} \times \frac{4}{9}$	**19.** $\frac{7}{12} \times 3\frac{1}{3}$	**31.** $9\frac{1}{11} \times 3\frac{3}{10}$
8. $\frac{5}{7} \times \frac{21}{25}$	**20.** $\frac{2}{3} \times 3\frac{1}{3}$	**32.** $5\frac{3}{5} \times 4\frac{2}{7}$
9. $14\frac{2}{3} \times 1\frac{7}{11}$	**21.** $12\frac{4}{5} \times 7\frac{5}{8}$	**33.** $6\frac{1}{4} \times 5\frac{3}{5}$
10. $6\frac{1}{4} \times \frac{24}{25}$	**22.** $17\frac{1}{2} \times 3\frac{5}{7}$	**34.** $18\frac{3}{4} \times 5\frac{1}{3}$
11. $4\frac{2}{5} \times \frac{5}{11}$	**23.** $3\frac{3}{5} \times 6\frac{1}{9}$	**35.** $19\frac{1}{2} \times 3\frac{1}{13}$
12. $14\frac{2}{7} \times 5\frac{3}{5}$	**24.** $4\frac{2}{7} \times 5\frac{3}{5}$	**36.** $14\frac{2}{5} \times 8\frac{1}{3}$

D. *Division*

Expressed as an improper fraction,

$$16\tfrac{1}{4} = ? \qquad 18\tfrac{1}{3} = ?$$

The multiplicative inverse of the divisor is _?_.

Complete the Example.

EXAMPLE

$$16\tfrac{1}{4} \div 18\tfrac{1}{3}$$
$$\tfrac{65}{4} \div \tfrac{55}{3}$$
$$\tfrac{65}{4} \times \tfrac{3}{55}$$

Divide: Write all answers in simplest form.

1. $7\frac{1}{7} \div \frac{5}{14}$
2. $6\frac{1}{4} \div \frac{15}{16}$
3. $17\frac{2}{3} \div \frac{10}{21}$
4. $4\frac{2}{7} \div \frac{10}{11}$
5. $\frac{5}{9} \div \frac{5}{6}$
6. $\frac{7}{12} \div \frac{1}{4}$
7. $\frac{15}{6} \div \frac{5}{8}$
8. $\frac{3}{4} \div \frac{9}{10}$
9. $1\frac{5}{9} \div 4\frac{1}{5}$
10. $3\frac{1}{2} \div 7\frac{1}{7}$
11. $2\frac{1}{8} \div 1\frac{17}{64}$
12. $6\frac{2}{5} \div 1\frac{3}{7}$
13. $7\frac{1}{2} \div 6\frac{1}{4}$
14. $1\frac{1}{32} \div 2\frac{4}{9}$
15. $15\frac{7}{8} \div 1\frac{5}{16}$
16. $7\frac{4}{7} \div 2\frac{1}{5}$
17. $2\frac{3}{4} \div 2\frac{1}{5}$
18. $\frac{5}{6} \div 2\frac{1}{12}$
19. $8\frac{1}{8} \div 3\frac{3}{4}$
20. $7\frac{1}{2} \div 2\frac{1}{3}$
21. $16\frac{3}{5} \div 6\frac{7}{10}$

E. *Addition of signed numbers*

Rules:

a. To add two numbers with like signs, add their absolute values and give the sum the common sign.

b. To add two numbers with unlike signs, subtract the absolute value of the smaller from the absolute value of the larger, and give the difference the sign of the larger.

EXAMPLES

1. $^{+}25 + (^{+}35) = |{}^{+}25| + |{}^{+}35| = {}^{+}(25 + 35) = {}^{+}(60) = {}^{+}60$

2. $^{-}17 + (^{-}53) = |{}^{-}17| + |{}^{-}53| = {}^{-}(17 + 53) = {}^{-}(70) = {}^{-}70$

3. $^{+}18 + (^{-}21) = |{}^{-}21| - |{}^{+}18| = {}^{-}(21 - 18) = {}^{-}(3) = {}^{-}3$

4. $^{-}41 + (^{+}18) = |{}^{-}41| - |{}^{+}18| = {}^{-}(41 - 18) = {}^{-}(23) = {}^{-}23$

Verify these Examples by using the number line.

Add:

1. $^{+}8 + (^{+}6)$
2. $^{+}15 + (^{-}19)$
3. $^{-}16 + (^{+}7)$
4. $^{-}24 + (^{-}25)$
5. $^{+}21 + (^{-}9)$
6. $^{+}36 + (^{+}40)$
7. $^{+}13 + (^{-}27)$
8. $^{-}18 + (^{-}16)$
9. $^{-}15 + (^{-}7)$

10. $\begin{array}{r} ^{+}28 \\ \underline{^{-}39} \end{array}$

11. $\begin{array}{r} ^{+}75 \\ \underline{^{+}96} \end{array}$

12. $\begin{array}{r} ^{-}59 \\ \underline{^{+}48} \end{array}$

13. $\begin{array}{r} ^{-}91 \\ \underline{^{-}75} \end{array}$

14. $\begin{array}{r} ^{+}43 \\ \underline{^{-}49} \end{array}$

15. $\begin{array}{r} ^{-}85 \\ \underline{^{-}96} \end{array}$

16. $\begin{array}{r} ^{+}47 \\ \underline{^{+}45} \end{array}$

17. $\begin{array}{r} ^{-}56 \\ \underline{^{-}62} \end{array}$

18. $\begin{array}{r} ^{-}25 \\ \underline{^{+}34} \end{array}$

F. *Subtraction of signed numbers*

Subtraction of signed numbers is most easily performed as an equivalent addition.

Rule: Add to the minuend the additive inverse of the subtrahend using the rules for adding signed numbers.

EXAMPLES

1. $^{+}5 - (^{+}3) = {}^{+}5 + (^{-}3)$
   $= {}^{+}2$
2. $^{-}18 - (^{+}19) = {}^{-}18 + (^{-}19)$
   $= {}^{-}37$
3. $^{+}17 - (^{+}21) = {}^{+}17 + (^{-}21)$
   $= {}^{-}4$
4. $^{-}9 - (^{-}12) = {}^{-}9 + (^{+}12)$
   $= {}^{+}3$

Verify these examples by using the number line.

Subtract:

1. $^{+}13 - (^{-}15)$
2. $^{+}18 - (^{+}15)$
3. $^{-}14 - (^{+}19)$
4. $^{-}17 - (^{-}30)$
5. $^{+}21 - (^{+}22)$
6. $^{+}18 - (^{-}41)$
7. $^{-}22 - (^{-}16)$
8. $^{-}47 - (^{-}50)$
9. $^{+}6 - (^{+}5)$
10. $^{-}19 - (^{+}11)$
11. $^{+}21 - (^{+}25)$
12. $^{+}35 - (^{-}16)$
13. $^{-}22 - (^{-}18)$
14. $^{-}27 - (^{+}30)$
15. $^{+}36 - (^{+}37)$

16. $\begin{array}{r} ^{+}35 \\ \underline{-\ (^{-}61)} \end{array}$

17. $\begin{array}{r} ^{+}33 \\ \underline{-\ (^{+}51)} \end{array}$

18. $\begin{array}{r} ^{-}47 \\ \underline{-\ (^{-}62)} \end{array}$

19. $\begin{array}{r} ^{+}28 \\ \underline{-\ (^{+}30)} \end{array}$

20. $\begin{array}{r} ^{-}21 \\ \underline{-\ (^{-}25)} \end{array}$

21. $\begin{array}{r} ^{-}23 \\ \underline{-\ (^{-}17)} \end{array}$

22. $\begin{array}{r} ^{-}37 \\ \underline{-\ (^{+}51)} \end{array}$

23. $\begin{array}{r} ^{+}42 \\ \underline{-\ (^{+}36)} \end{array}$

24. $\begin{array}{r} ^{+}45 \\ \underline{-\ (^{-}27)} \end{array}$

25. $\begin{array}{r} ^{+}53 \\ \underline{-\ (^{+}46)} \end{array}$

26. $\begin{array}{r} ^{+}41 \\ \underline{-\ \ (^{+}8)} \end{array}$

27. $\begin{array}{r} ^{-}31 \\ \underline{-\ (^{+}17)} \end{array}$

A. *Add:* Write all answers in simplest form.

1. $7\frac{4}{5} + 8\frac{3}{4}$

2. $4\frac{5}{6} + 3\frac{7}{8}$

3. $6\frac{1}{6} + 5\frac{2}{3}$

4. $2\frac{5}{6} + \frac{9}{10}$

5. $\begin{array}{r} 10\frac{3}{4} \\ 4\frac{1}{12} \\ \underline{2\frac{5}{6}} \end{array}$

6. $\begin{array}{r} 3\frac{7}{12} \\ 14\frac{7}{8} \\ \underline{15\frac{2}{3}} \end{array}$

7. $\begin{array}{r} 1\frac{2}{5} \\ 3\frac{5}{8} \\ \underline{2\frac{3}{4}} \end{array}$

B. *Subtract:* Write all answers in simplest form.

1. $\begin{array}{r} 7 \\ \underline{4\frac{2}{5}} \end{array}$

2. $\begin{array}{r} 3\frac{15}{16} \\ \underline{\frac{3}{5}} \end{array}$

3. $\begin{array}{r} 17\frac{3}{4} \\ \underline{6\frac{3}{8}} \end{array}$

4. $\begin{array}{r} 11\frac{7}{9} \\ \underline{3\frac{5}{6}} \end{array}$

5. $\begin{array}{r} 12\frac{5}{9} \\ \underline{7\frac{1}{2}} \end{array}$

6. $\begin{array}{r} 17\frac{1}{6} \\ \underline{9\frac{1}{4}} \end{array}$

7. $\begin{array}{r} 5\frac{7}{12} \\ \underline{3\frac{8}{9}} \end{array}$

8. $\begin{array}{r} 9\frac{1}{2} \\ \underline{7\frac{1}{8}} \end{array}$

9. $\begin{array}{r} 14\frac{1}{5} \\ \underline{10\frac{5}{6}} \end{array}$

10. $\begin{array}{r} 3\frac{3}{5} \\ \underline{\frac{9}{10}} \end{array}$

11. $\begin{array}{r} 14\frac{7}{8} \\ \underline{9\frac{3}{4}} \end{array}$

12. $\begin{array}{r} 25\frac{2}{5} \\ \underline{16\frac{1}{2}} \end{array}$

C. *Multiply:*

1. $36 \times 1\frac{5}{6}$

2. $9 \times \frac{2}{3}$

3. $18 \times \frac{7}{9}$

4. $3\frac{3}{4} \times 1\frac{3}{5}$

5. $1\frac{5}{8} \times \frac{11}{12}$

6. $\frac{9}{14} \times 2\frac{7}{18}$

7. $14\frac{2}{3} \times 7\frac{1}{2}$

8. $6\frac{1}{4} \times \frac{12}{25}$

9. $33\frac{1}{3} \times \frac{12}{25}$

D. *Divide:*

1. $6\frac{1}{8} \div 7$

2. $39 \div 1\frac{5}{8}$

3. $18 \div \frac{3}{4}$

4. $24 \div \frac{7}{16}$

5. $7\frac{3}{5} \div 3\frac{1}{6}$

6. $\frac{13}{15} \div 8\frac{2}{3}$

7. $\frac{3}{4} \div 15$

8. $\frac{5}{12} \div 25$

9. $\frac{17}{12} \div \frac{7}{8}$

10. $38\frac{1}{2} \div \frac{11}{16}$

11. $10\frac{1}{2} \div \frac{3}{8}$

12. $19\frac{1}{2} \div \frac{13}{16}$

E. *Add:*

1. ${}^{+}5 + ({}^{-}4)$

2. ${}^{-}18 + ({}^{+}8)$

3. ${}^{-}6 + ({}^{+}6)$

4. ${}^{+}12 + ({}^{+}7)$

5. ${}^{-}9 + ({}^{-}9)$

6. ${}^{-}3 + ({}^{-}18)$

F. *Subtract:*

1. ${}^{+}14 - ({}^{+}12)$

2. ${}^{+}9 - ({}^{+}15)$

3. ${}^{+}15 - ({}^{+}9)$

4. ${}^{-}7 - ({}^{-}7)$

5. ${}^{-}7 - ({}^{+}7)$

6. ${}^{-}25 - ({}^{-}20)$

7. ${}^{+}18 - ({}^{+}6)$

8. ${}^{+}27 - ({}^{-}13)$

9. ${}^{-}19 - ({}^{+}15)$

10. ${}^{-}32 - ({}^{-}28)$

11. ${}^{+}21 - ({}^{+}15)$

12. ${}^{+}27 - ({}^{-}9)$

A. *Writing a per cent as a decimal*

The symbol % means "× .01." Therefore, to write a per cent as a decimal, you omit the per cent symbol and move the decimal point two places to the left, inserting zeros if needed.

EXAMPLES

23.5% = 0.235
3% = 0.03
175% = 1.75

Write each of the following as a decimal.

1. 86%	**11.** 0.015%	**21.** 1.1%
2. 15%	**12.** 108.4%	**22.** 13%
3. 120%	**13.** 95%	**23.** 0.07%
4. 75%	**14.** 2%	**24.** 2.5%
5. 43.2%	**15.** 1.25%	**25.** 62.5%
6. 750%	**16.** 57.1%	**26.** 87.5%
7. 93%	**17.** 123.2%	**27.** 136.1%
8. 16.3%	**18.** 4.1%	**28.** 400%
9. 5.07%	**19.** 19.75%	**29.** 250%
10. 19%	**20.** 0.1%	**30.** 175%

B. *Writing a decimal as a per cent*

Writing a decimal as a per cent is the inverse of the process used in A above. Move the decimal point two places to the right, and affix the per cent symbol. Study the Examples.

EXAMPLES

0.035 = 3.5%
0.875 = 87.5%
5.25 = 525%

Write each of the following as a per cent.

1. 0.35	**11.** 0.0025	**21.** 0.7
2. 0.03	**12.** 0.845	**22.** 6.25
3. 0.33	**13.** 3.52	**23.** 0.075
4. 0.046	**14.** 8.23	**24.** 0.18
5. 0.095	**15.** 6.03	**25.** 2.5
6. 0.012	**16.** 7.05	**26.** 0.27
7. 0.501	**17.** 0.5	**27.** 1.9
8. 0.233	**18.** 0.025	**28.** 0.4
9. 0.003	**19.** 3.07	**29.** 0.036
10. 0.0052	**20.** 0.17	**30.** 2.06

C. *Writing a fraction as a per cent*

To write a fraction as $n\%$, write the proportion equating the fraction to $\frac{n}{100}$, and solve the proportion.

EXAMPLE

$\frac{5}{9} = n\%$ or $\frac{5}{9} = \frac{n}{100}$

Therefore, $500 = 9n$ $\qquad p = x \times y$

$500 \div 9 = n$ $\qquad p \div x = y$

$55.55 = n$

Then: $\frac{5}{9} = \frac{55.55}{100}$

$= 55.55\%$ or 55.6% (to the nearest tenth of 1%)

Express each of the following as a per cent rounded to the nearest tenth of 1%.

1. $\frac{1}{2}$	**6.** $\frac{3}{5}$	**11.** $\frac{13}{16}$	**16.** $\frac{3}{8}$
2. $\frac{3}{4}$	**7.** $\frac{3}{7}$	**12.** $\frac{1}{40}$	**17.** $\frac{27}{20}$
3. $\frac{5}{8}$	**8.** $\frac{4}{5}$	**13.** $\frac{3}{20}$	**18.** $\frac{34}{15}$
4. $\frac{2}{5}$	**9.** $\frac{7}{16}$	**14.** $\frac{7}{25}$	**19.** $\frac{23}{20}$
5. $\frac{19}{20}$	**10.** $\frac{4}{9}$	**15.** $\frac{1}{5}$	**20.** $\frac{14}{9}$

D. *Finding a missing term in a percentage statement*

With the per cent expressed as a fraction whose denominator is 100, write the proportion equating the two ratios. Then solve for the missing term. Study the Examples below. Then find the value of N in each of the Exercises on page 472.

EXAMPLES

1. N is 27% of 65.

$$\frac{N}{65} = \frac{27}{100}$$
$$100N = 1755$$
$$\text{Then } N = 17.55$$

2. $N\%$ of 56 is 42.

$$\frac{N}{100} = \frac{42}{56}$$
$$56N = 4200$$
$$\text{Then } N = 75$$

3. 85% of N is 46.75

$$\frac{85}{100} = \frac{46.75}{N}$$
$$85N = 4675$$
$$\text{Then } N = 55$$

1. 60% of 85 is N
2. 37 is $N\%$ of 185
3. $N\%$ of 84 is 21
4. 28% of N is 14
5. 57 is $N\%$ of 95
6. 45 is $N\%$ of 50
7. 90% of 162 is N
8. 37.5% of N is 57
9. 2.5 is $N\%$ of 500
10. 66 is $N\%$ of 220
11. 25% of N is 125
12. 52 is 130% of N
13. 15 is 20% of N
14. 3% of 6500 is N
15. $N\%$ of 360 is 72
16. 75 is $N\%$ of 375
17. 23% of 60 is N
18. 3% of N is 9
19. 11% of N is 847
20. 98 is $N\%$ of 70
21. 54% of 160 is N
22. N is 45% of 630
23. 84 is $N\%$ of 1200
24. $N\%$ of 68 is 51
25. 120% of N is 84
26. 48 is 75% of N
27. 0.5% of 48 is N
28. 169 is $N\%$ of 65
29. 45 is $N\%$ of 80
30. 4.5% of N is 36

PRACTICE TEST

A. Express each of the following as a per cent.

1. 0.39
2. 0.875
3. 2.05
4. 0.008
5. 2.22
6. 4.50

B. Express each of the following as a per cent rounded to the nearest tenth of 1%.

1. $\frac{1}{4}$
2. $\frac{3}{5}$
3. $\frac{15}{16}$
4. $\frac{19}{20}$
5. $\frac{5}{8}$
6. $\frac{13}{25}$
7. $\frac{9}{5}$
8. $\frac{16}{3}$
9. $\frac{16}{9}$
10. $\frac{3}{200}$
11. $\frac{12}{81}$
12. $\frac{4}{500}$

C. Express each of the following as a decimal.

1. 55%
2. 2.3%
3. 143%
4. 0.2%
5. 112.5%
6. 6.2%

D. Find the value of N in each of the following:

1. 35% of 16 is N
2. N is 80% of 40
3. 18.7% of 124 is N
4. N is 15.9% of 80
5. 195% of 140 is N
6. 3.2% of 600 is N
7. $N\%$ of 40 is 18
8. 3.6 is $N\%$ of 4.8
9. 0.4 is $N\%$ of 1.6
10. $N\%$ of 20 is 16

11. 24 is $N\%$ of 6
12. 75 is $N\%$ of 500
13. 85 is 68% of N
14. 40% of N is 32
15. 625 is 2.5% of N
16. 340 is 17% of N
17. 80 is 160% of N
18. 16 is 0.5% of N
19. 5.7 is $N\%$ of 7.6
20. 0.3% of 66 is N
21. N is 125% of 36
22. 2.5% of 160 is N
23. 0.9% of 1200 is N
24. N is 183.7% of 180
25. N is 15% of 1400
26. 45 is $N\%$ of 225
27. 37.5% of N is 27
28. 0.2% of N is 44
29. 13.5 is $N\%$ of 15
30. $N\%$ of 1200 is 3
31. 840 is $N\%$ of 120
32. 135% of 60 is N
33. 36 is $N\%$ of 400
34. 7.5% of N is 12
35. 1.3% of 600 is N
36. 360 is 150% of N
37. 48 is $N\%$ of 32
38. $N\%$ of 7.5 is 1.5
39. 75% of N is 9.6
40. 24 is $N\%$ of 9600

PRACTICE EXERCISES

A. *To express a per cent as a decimal*

> *Rule:* To express a per cent as a decimal, move the decimal point two places to the left, inserting zeros as necessary, and omit the per cent sign.

Write each of the following as a decimal.

1. 14%
2. 18%
3. 25%
4. 40%
5. 98%
6. 2%
7. 9%
8. 1.9%
9. 3.43%
10. 16.25%
11. 100%
12. 3.1%
13. 65.5%
14. 121%
15. 0.5%
16. 1.05%
17. 2.5%
18. 200%

B. *Writing a decimal as a per cent*

> *Rule:* To write a decimal as a per cent, move the decimal point two places to the right, annexing zeros if needed. Affix the per cent sign (%).

Write each of the following as a per cent.

1. 0.35	**8.** 0.238	**15.** 3.03
2. 0.98	**9.** 0.305	**16.** 21.13
3. 0.882	**10.** 0.8032	**17.** 0.0062
4. 0.046	**11.** 0.0025	**18.** 1.75
5. 0.09	**12.** 0.875	**19.** 0.05
6. 0.07	**13.** 3.58	**20.** 0.006
7. 0.564	**14.** 2.28	**21.** 4.5

C. *Writing a fraction as a per cent*

To write a fraction as a per cent, set up and solve the proportion:

$$\frac{a}{b} = \frac{n}{100}$$

where $\frac{a}{b}$, $b \neq 0$, names the fractional number.

EXAMPLES

1. Write $\frac{5}{8}$ as a per cent.

$\frac{5}{8} = n\%$ or $\frac{5}{8} = \frac{n}{100}$

Then $500 = 8n$ or $8n = 500$ $\qquad x \times y = p$

$n = 500 \div 8$ $\qquad x = p \div y$

$n = 62.5$

Then $\frac{5}{8} = \frac{62.5}{100}$

and $\frac{5}{8} = 62.5\%$

2. Write $\frac{3}{7}$ as a per cent.

If necessary, carry the division out two places beyond the decimal point when finding n. Then round the per cent to the nearest tenth of 1%.

$\frac{3}{7} = n\%$ or $\frac{3}{7} = \frac{n}{100}$

Then $300 = 7n$ or $7n = 300$ $\qquad x \times y = p$

$n = 300 \div 7$ $\qquad x = p \div y$

$n = 42.85$

Then $\frac{3}{7} = \frac{42.85}{100}$

and $\frac{3}{7} = 42.85\%$ or 42.9%

Write each of the following as a per cent rounded to the nearest tenth of 1%, where necessary.

1. $\frac{3}{5}$	**7.** $\frac{3}{16}$	**13.** $\frac{1}{6}$	**19.** $\frac{2}{11}$
2. $\frac{5}{8}$	**8.** $\frac{7}{24}$	**14.** $\frac{7}{30}$	**20.** $\frac{3}{17}$
3. $\frac{1}{2}$	**9.** $\frac{27}{100}$	**15.** $\frac{7}{8}$	**21.** $\frac{19}{100}$
4. $\frac{4}{5}$	**10.** $\frac{9}{16}$	**16.** $\frac{5}{6}$	**22.** $\frac{39}{500}$
5. $\frac{11}{50}$	**11.** $\frac{3}{8}$	**17.** $\frac{9}{5}$	**23.** $\frac{16}{3}$
6. $\frac{7}{16}$	**12.** $\frac{7}{12}$	**18.** $\frac{13}{4}$	**24.** $\frac{37}{1000}$

D. *Finding a missing part in a percentage statement*

Write the proportion with the per cent expressed as a fraction equal to the ratio between the other two numbers. Solve the proportion.

Study the Examples to see how the unknown numbers are expressed as variables and how the proportions are set up and solved.

Then solve for N in each of the Exercises on page 476.

EXAMPLES

1. $N\%$ of 24 is 18

$$N\% = \frac{N}{100} \qquad \text{Then } \frac{N}{100} = \frac{18}{24}$$
$$24N = 1800 \qquad x \times y = p$$
$$N = 1800 \div 24 \qquad x = p \div y$$
$$N = 75$$
$$\text{Then } N\% = 75\%$$

2. N is 15% of 45

$$15\% = \frac{15}{100} \qquad \text{Then } \frac{N}{45} = \frac{15}{100}$$
$$100N = 675 \qquad x \times y = p$$
$$N = 675 \div 100 \qquad x = p \div y$$
$$N = 6.75$$

3. 56% of N is 22.4

$$56\% = \frac{56}{100} \qquad \text{Then } \frac{56}{100} = \frac{22.4}{N}$$
$$56N = 2240 \qquad x \times y = p$$
$$N = 2240 \div 56 \qquad x = p \div y$$
$$N = 40$$

1. 40% of 75 is N
2. 37 is 20% of N
3. 76 is N% of 95
4. N% of 6.4 is 5.6
5. 87 is N% of 116
6. 125 is N% of 200
7. 90% of 56 is N
8. 37.5% of N is 72
9. 2.5 is N% of 750
10. 44 is N% of 110
11. N is 25% of 65
12. 25% of N is 65
13. 13.5 is 45% of N
14. 2% of 6400 is N
15. N% of 36 is 10.8
16. N% of 84 is 105
17. 28% of N is 56
18. 3% of N is 12
19. 41% of N is 16.4
20. 9.8 is N% of 7
21. 5.4% of 70 is N
22. N is 4.5% of 63
23. 3 is N% of 1200
24. N% of 68 is 85
25. 180% of N is 8.1
26. 49 is 1.75% of N
27. 0.5% of N is 48
28. 16.9 is N% of 65

E. *Solving proportions*

To solve a proportion, equate the product of the means to the product of the extremes. Then solve the equation.

EXAMPLE

Solve for n.

$$\frac{14}{n} = \frac{21}{24}$$
$$21n = 336$$
$$n = 16$$

Check:

$$\frac{14}{16} = \frac{21}{24}$$
$$\frac{7}{8} = \frac{7}{8}$$

1. $\frac{n}{8} = \frac{6}{16}$
2. $\frac{14}{n} = \frac{42}{48}$
3. $\frac{8}{12} = \frac{n}{18}$
4. $\frac{15}{16} = \frac{n}{100}$
5. $\frac{18}{21} = \frac{12}{n}$
6. $\frac{n}{100} = \frac{19}{20}$
7. $\frac{6}{18} = \frac{n}{3}$
8. $\frac{9}{12} = \frac{27}{n}$
9. $\frac{24}{25} = \frac{n}{100}$
10. $\frac{8}{12} = \frac{100}{n}$
11. $\frac{n}{100} = \frac{4}{16}$
12. $\frac{16}{n} = \frac{8}{9}$

A. Express each of the following as a decimal.

1. 78%
2. 19%
3. 17.5%
4. 2.3%
5. 0.25%
6. 0.4%
7. 169.2%
8. 425%
9. 0.003%

B. Express each of the following as a per cent.

1. 0.35
2. 0.95
3. 0.05
4. 0.375
5. 0.125
6. 0.1625
7. 1.5
8. 2.25
9. 12

C. Express each of the following as a per cent rounded to the nearest tenth of 1%.

1. $\frac{4}{25}$
2. $\frac{3}{5}$
3. $\frac{5}{16}$
4. $\frac{17}{40}$
5. $\frac{3}{200}$
6. $\frac{1}{50}$
7. $\frac{7}{4}$
8. $\frac{5}{2}$
9. $\frac{5}{3}$

D. Copy and solve for N.

1. 64% of 80 is N
2. 27 is N% of 36
3. 3.5% of 120 is N
4. 120% of 60 is N
5. 80% of N is 56
6. 219% of 56 is N

E. Solve for n.

1. $\frac{5}{8} = \frac{n}{100}$
2. $\frac{n}{6} = \frac{8}{12}$
3. $\frac{9}{n} = \frac{3}{4}$
4. $\frac{8}{15} = \frac{16}{n}$
5. $\frac{n}{12} = \frac{6}{8}$
6. $\frac{14}{16} = \frac{n}{24}$

PRACTICE EXERCISES

A. *Addition*

1.	2.	3.
494	456	444
756	206	693
380	393	501
871	735	259
502	938	537
645	577	358

4. 132
875
948
794
387
970

5. 295
783
254
849
593
560

6. 485
250
487
382
794
849

7. 393
581
489
219
948
307

8. $153.48
17.80
30.73
39.45
4.99
102.09

9. $ 58.23
3.63
.74
4.78
71.86
548.39

B. *Subtraction*

1. 400
273

2. 340
209

3. 300
217

4. 200
180

5. 570
277

6. 300
273

7. 705
656

8. 200
148

9. 480
402

10. 407
309

11. 420
311

12. 210
190

13. 105
70

14. 414
321

15. 518
420

16. 316
240

17. 106
59

18. 812
410

C. *Multiplication*

1. 221 × 30
2. 332 × 20
3. 218 × 20
4. 609 × 81
5. 207 × 18
6. 501 × 59
7. 419 × 206
8. 766 × 308
9. 23 × $64.45
10. 49 × $65.05
11. 49 × $72.50
12. 47 × $76.85
13. 82 × $76.06
14. 92 × $47.25
15. 68 × $85.80
16. 79 × $45.92
17. 83 × $59.50
18. 97 × $77.95
19. 83 × $74.20
20. 52 × $69.80
21. 30 × $75.56
22. 206 × $125.35
23. 607 × $36.08
24. 109 × $50.80

D. *Division*

When dividing with dollars and cents, the quotient is usually expressed to the nearest cent. In Example 1, the remainder is less than one-half of the divisor. Therefore, the answer is rounded to \$.45. In Example 2, the remainder is not considered, because the division was carried out an additional decimal place and will then be rounded to the nearest cent. You may use either procedure.

EXAMPLES

1.
```
         .45
   25)$11.29
       10 0
        1 29
        1 25
           4
```

2.
```
         .451
   25)$11.290
       10 0
        1 29
        1 25
           40
           25
           15
```

Divide: Round to the nearest hundredth or to the nearest cent.

1. $4575 \div 40$
2. $6556 \div 56$
3. $2116 \div 37$
4. $6644 \div 42$
5. $9666 \div 24$
6. $9256 \div 184$
7. $6284 \div 194$
8. $6418 \div 116$
9. $8737 \div 174$
10. $9848 \div 196$
11. $15{,}248 \div 188$
12. $15{,}360 \div 264$
13. \$40.86 ÷ 16
14. \$24.59 ÷ 30
15. \$59.03 ÷ 54
16. \$70.81 ÷ 42
17. \$73.98 ÷ 65
18. \$86.26 ÷ 52
19. \$65.85 ÷ 34
20. \$89.09 ÷ 37
21. \$64.48 ÷ 76
22. \$49.50 ÷ 40
23. \$20.43 ÷ 38
24. \$79.02 ÷ 53

E. Find the value for each of the following:

1. $4\frac{5}{8} - (^{+}3\frac{1}{2})$
2. $8\frac{1}{3} + (^{-}11\frac{2}{3})$
3. $6\frac{5}{9} + (^{+}3\frac{1}{2})$
4. $5\frac{3}{16} - (^{+}6\frac{5}{8})$
5. $6\frac{2}{3} - (^{-}5\frac{1}{3})$
6. $5\frac{3}{16} + (^{-}4\frac{9}{16})$
7. $7\frac{8}{9} - (^{+}9\frac{2}{3})$
8. $9\frac{5}{6} + (^{-}12\frac{1}{2})$
9. $6\frac{4}{5} + (^{+}5\frac{9}{10})$
10. $7\frac{3}{8} + (^{-}6\frac{7}{8})$
11. $8\frac{3}{5} - (^{+}9\frac{7}{10})$
12. $3\frac{1}{2} - (^{-}7\frac{3}{4})$
13. $7\frac{2}{5} + (^{+}9\frac{1}{2})$
14. $7\frac{3}{4} - (^{+}8\frac{1}{2})$
15. $5\frac{3}{4} - (^{-}8\frac{1}{4})$

F. *Subtraction with signed numbers*

With signed numbers the process of subtraction is replaced by an equivalent addition.

Rule: Add to the minuend the additive inverse of the subtrahend using the rules for adding signed numbers.

EXAMPLES

1. $^{-}9 - (^{+}6)$
   $= {}^{-}9 + (^{-}6) = {}^{-}15$
2. $^{+}22.6 - (^{+}16.5)$
   $= {}^{+}22.6 + (^{-}16.5) = {}^{+}6.1$
3. $^{+}8\frac{5}{8} - (^{+}9\frac{3}{4})$
   $= {}^{+}8\frac{5}{8} + (^{-}9\frac{3}{4}) = {}^{-}1\frac{1}{8}$

Subtract:

1. $^{+}16 - (^{-}4)$
2. $^{-}18 - (^{-}3)$
3. $^{-}13 - (^{-}4)$
4. $^{+}20 - (^{+}24)$
5. $^{-}24 - (^{+}4)$
6. $^{+}36 - (^{+}54)$
7. $^{-}22.4 - (^{-}2.3)$
8. $^{+}14.1 - (^{-}12.7)$
9. $^{-}26.2 - (^{-}20.8)$
10. $^{-}28.1 - (^{+}30.5)$
11. $^{+}20.2 - (^{+}25.9)$
12. $^{-}18.0 - (^{-}19.5)$
13. $^{-}15\frac{1}{2} - (^{-}5\frac{2}{3})$
14. $^{+}12\frac{1}{2} - (^{+}6\frac{3}{4})$
15. $^{-}6\frac{2}{5} - (^{-}2\frac{3}{10})$
16. $^{-}9\frac{1}{2} - (^{-}13\frac{3}{8})$
17. $^{+}26\frac{5}{6} - (^{+}37\frac{3}{4})$
18. $^{-}36\frac{4}{5} - (^{+}12\frac{9}{10})$

G. *Solving conditional statements*

Many conditional statements may be solved by using the addend-addend-sum relationship or the factor-factor-product relationship.

EXAMPLES

Solve for n.

1. $n - 13 = 27$
   $n = 13 + 27$
   $n = 40$

   $s - a = b$
   $s = a + b$

2. $\frac{3n}{5} = 27$
   $n = 27 \div \frac{3}{5}$
   $n = 27 \times \frac{5}{3}$
   $n = 45$

   $xy = p$
   $x = p \div y$

3. $\frac{24}{n} = 3$
   $n = 24 \div 3$
   $n = 8$

   $\frac{p}{y} = x$
   $y = p \div x$

Solve for n.

1. $n + 12 = 40$

2. $8n = 72$

3. $16 \div n = 2$

4. $n - 18 = 21$

5. $n \div 12 = 6$

6. $16 - n = 5$

7. $\frac{5n}{8} = 25$

8. $28 = n + 17$

9. $13n = 91$

10. $54 \div n = 6$

11. $n \div 17 = 5$

12. $56 = n + 24$

13. $27 = 3n$

14. $19 = n - 27$

15. $5 = \frac{80}{n}$

16. $n - 25 = 15$

PRACTICE TEST

A. *Add:*

1.	**2.**	**3.**	**4.**
\$3145.43	\$ 8.51	\$ 7.25	\$9039.56
16.84	2150.43	2083.17	83.45
117.55	16.72	56.45	421.31
44.30	114.59	133.25	3.52
.56	3145.83	3075.19	6051.63

B. *Subtract:*

1.	**3.**	**5.**	**7.**
\$119.00	\$600.00	\$300.00	\$116.32
88.45	19.35	100.50	85.44

2.	**4.**	**6.**	**8.**
\$90.70	\$64.45	\$866.50	\$1000.00
56.05	30.65	109.98	200.05

C. *Multiply:*

1. 704×115

2. 516×205

3. 310×206

4. $156 \times \$175.53$

5. $205 \times \$1255.00$

6. $119 \times \$185.75$

D. *Divide:* Round to the nearest hundredth or to the nearest cent.

1. \$32.50 ÷ 39

2. \$90.95 ÷ 87

3. \$86.45 ÷ 65

4. 1943 ÷ 225

5. 11678 ÷ 706

6. 21375 ÷ 295

E. Find the value for each of the following:

1. $^{+}4\frac{3}{5} + (^{+}8\frac{1}{8})$

2. $^{+}9\frac{5}{8} + (^{-}2\frac{1}{2})$

3. $^{+}3\frac{2}{5} - (^{+}8\frac{3}{10})$

4. $^{+}8\frac{1}{3} - (^{-}7\frac{5}{9})$

5. $^{+}6\frac{7}{8} - (^{+}9\frac{3}{4})$

6. $^{+}4\frac{5}{16} + (^{-}3\frac{5}{8})$

F. *Subtract:*

1. $^{+}12 - (^{-}15)$
2. $^{-}18 - (^{-}3)$
3. $^{+}16.7 - (^{+}18.5)$
4. $^{-}25.3 - (^{+}5.7)$
5. $^{+}24\frac{2}{3} - (^{-}16\frac{3}{4})$
6. $^{-}36\frac{1}{2} - (^{+}6\frac{7}{8})$

G. Solve for n.

1. $n + 18 = 25$
2. $8n = 24$
3. $18 - n = 11$
4. $\frac{16n}{3} = 48$
5. $\frac{n}{13} = 6$
6. $n - 15 = 17$
7. $\frac{27}{n} = 3$

Tables

LINEAR MEASURE

12 inches (in.)	= 1 foot (ft.)
3 feet	= 1 yard (yd.)
$5\frac{1}{2}$ yards, or $16\frac{1}{2}$ feet	= 1 rod (rd.)
320 rods, 1760 yards, or 5280 feet	= 1 mile (mi.)

SQUARE MEASURE

144 square inches (sq. in.)	= 1 square foot (sq. ft.)
9 square feet	= 1 square yard (sq. yd.)
$30\frac{1}{4}$ square yards	= 1 square rod (sq. rd.)
160 square rods, or 43,560 square feet	= 1 acre (A.)
640 acres	= 1 square mile (sq. mi.)

CUBIC MEASURE

1728 cubic inches (cu. in.)	= 1 cubic foot (cu. ft.)
27 cubic feet	= 1 cubic yard (cu. yd.)

LIQUID MEASURE

3 teaspoons	= 1 tablespoon
16 tablespoons	= 1 cup
2 cups	= 1 pint
1 pint	= 16 ounces (liquid)
2 pints	= 1 quart
4 quarts	= 1 gallon
7.5 gallons	$\approx$ 1 cubic foot

DRY MEASURE

2 pints (pt.)	= 1 quart (qt.)
8 quarts	= 1 peck (pk.)
4 pecks	= 1 bushel (bu.)

AVOIRDUPOIS WEIGHT

16 ounces (oz.) = 1 pound (lb.)
100 pounds = 1 hundredweight (cwt.)
20 hundredweight, or 2000 pounds = 1 ton (T.)

ANGLES AND ARCS

60 seconds (″) = 1 minute (′)
60 minutes = 1 degree (°)
90 degrees = 1 right angle
360 degrees of arc = 1 circumference
360 degrees of angle = 1 complete rotation

TIME

60 seconds (sec.) = 1 minute (min.)
60 minutes = 1 hour (hr.)
24 hours = 1 day (da.)
7 days = 1 week (wk.)
365 days = 1 common year (yr.)
366 days = 1 leap year
12 months (mo.) = 1 year
360 days = 1 commercial year

METRIC LINEAR UNITS

1 millimeter (mm.) = 0.001 meter (m.)
1 centimeter (cm.) = 0.01 meter
1 decimeter (dm.) = 0.1 meter
1 dekameter (dkm.) = 10 meters
1 hectometer (hm.) = 100 meters
1 kilometer (km.) = 1000 meters

METRIC MEASURES OF CAPACITY

10 milliliters = 1 centiliter (cl.)
10 centiliters = 1 deciliter (dl.)
10 deciliters = 1 liter (l.)
10 liters = 1 dekaliter (dkl.)
10 dekaliters = 1 hectoliter (hl.)
10 hectoliters = 1 kiloliter (kl.)

METRIC WEIGHT

10 milligrams (mg.)	= 1 centigram (cg.)
10 centigrams	= 1 decigram (dg.)
10 decigrams	= 1 gram (g.)
10 grams	= 1 dekagram (dkg.)
10 dekagrams	= 1 hectogram (hg.)
10 hectograms	= 1 kilogram (kg.)

COMMON EQUIVALENTS (APPROXIMATE)

1 bu.	$\approx$ 2150 cu. in. or $1\frac{1}{4}$ cu. ft.
1 gal.	$\approx$ 231 cu. in.
1 cu. ft.	$\approx$ $7\frac{1}{2}$ gal.
1 cu. ft. water	$\approx$ 62.5 lb.
1 gal. water	$\approx$ $8\frac{1}{3}$ lb.

METRIC EQUIVALENTS

Linear

1 inch	= 2.54 centimeters
1 yard	= 0.9144 meter
1 mile	$\approx$ 1.609 kilometers
1 centimeter	$\approx$ 0.39 inch
1 meter	$\approx$ 39.37 inches, or 1.1 yard
1 kilometer	$\approx$ 0.62 mile

Liquid

1 liter	$\approx$ 1.056 quarts
1 quart	$\approx$ 0.95 liters
1 gallon	$\approx$ 3.8 liters

Dry

1 liter	$\approx$ 0.91 quarts
1 quart	$\approx$ 1.1 liters
1 hectoliter	$\approx$ 2.8 bushels

Weight

1 gram	$\approx$ 0.035 oz.
1 kg.	$\approx$ 2.2 lb.
1 MT	$\approx$ 2204.6 lb.
1 oz.	$\approx$ 28.4 g.
1 lb.	$\approx$ 0.453 kg.

Glossary of Mathematical Terms

The explanations in this glossary are intended as useful clarifications of the terms listed. They are not to be interpreted as precise definitions.

Absolute value The value of a number independent of its sign

Acute angle An angle whose measure is less than 90°

Acute triangle A triangle all of whose angles are acute

Addends Numbers to be added

Additive inverse If the sum of two numbers is zero, each is the additive inverse of the other.

Altitude of a triangle The segment from a vertex perpendicular to the line containing the opposite side

Angle The figure formed by two rays with a common endpoint

Approximate number A number that expresses nearly, but not exactly, a measurement or count

Arc A part of a circle

Arithmetic progression A sequence of numbers with the same difference between consecutive terms

Area The number of square units of surface in a region

Associative property The way in which the addends (or factors) are grouped does not affect the sum (or product); i.e., $(a + b) + c = a + (b + c)$; $(a \times b) \times c = a \times (b \times c)$

Average The sum of a set of numbers divided by the number of elements in the set

Bar graph A graph in which relative values are represented by the lengths of the bars

Base The line or surface on which a plane or solid figure rests

Base of a number system The number used as a factor in an exponential system of numeration

Bisect To divide into two equal parts

Capacity The measure of contents of a container

Cardinal numbers Numbers used to tell how many elements are in a set

Center of a circle The point which is equidistant from all points of the circle

Chord A segment whose endpoints are on a circle

Circle A closed curve, each of whose points is equally distant from its center

Circle graph A graph using sectors of a circle to show how the whole of anything is divided up

Circumference The measure of a circle

Collinear Lying in the same straight line

Combination A set considered without reference to the order of arrangement of its elements

Common denominator A number divisible by each of the denominators of a set of fractions

Commutative property The order in which the addends (or factors) are added (or multiplied) does not affect the sum (or product); i.e., $a + b = b + a$, $a \times b = b \times a$

Composite number A number with factors other than itself and 1

Conditional statement A mathematical statement that is neither true nor false; an open sentence; i.e., $x + 5 = 7$

Congruent figures Two figures are congruent if, when one is placed on the other, they fit exactly.

Decimal A fraction whose denominator is some power of 10, expressed with a decimal point to the left of the tenths place

Decimal system of numeration The writing of numerals using powers of 10 as place value

Degree A unit of measure for angles

Denominator The number indicated below the bar in a fraction; the number of parts into which the whole was divided

Dependent events Events in which the performance of one affects the outcome of the other

Diagonal A segment, other than a side, connecting two vertices

Diameter A chord which contains the center of a circle

Difference The result of subtraction

Digit Any of the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

Dimension The length of a segment in a given direction, as length, width, or depth

Distributive property A property of our number system

Dividend The number to be divided

Element A member of a set

Empty set The set which contains no elements

Equation A statement that two expressions name the same number or the same set

Equilateral polygon A polygon whose sides all have the same measure

Equivalent fractions Fractions naming the same number

Error of measurement One-half of the smallest unit used in the measurement

Estimating Finding an approximate answer without precise computation or measurement

Evaluate To find the value of an expression

Exponent A number that tells how many times another number is to be taken as a factor

Extremes The first and fourth terms of a proportion

Factorial n factorial, $n!$ is $n(n-1)(n-2)\ldots(1)$

Factorization The process of naming a number as a product; i.e., the factorizations of 30 are $2 \times 3 \times 5$, 1×30, 2×15, 3×10, 5×6

Finite set A set in which the number of elements can be described by a whole number

Formula A general rule stated as a mathematical sentence

Fraction The indicated division of an integer by a non-zero integer

Fractional numbers of arithmetic The positive rational numbers and zero

Graph A pictorial representation of numerical data

Hexagon A polygon of six sides

Horizontal Parallel to the horizon or to the ground

Improper fraction A fraction whose numerator is greater than its denominator

Independent events Events are independent if the performance of one does not affect the outcome of the other.

Indirect measurement of distances Measurement of distances between points inaccessible to one another, or where it is impractical to use a measuring instrument directly

Infinite Extending without limit

Intersection of sets The set containing all elements which are common to the given sets

Integers The set of numbers, $\{\ldots, {}^{-}3, {}^{-}2, {}^{-}1, 0, {}^{+}1, {}^{+}2, \ldots\}$

Inverse operations Operations that undo one another, as addition and subtraction, multiplication and division

Isosceles triangle A triangle in which two sides have the same measure

Like fractions Fractions having the same denominator

Linear measurement Measures of distance or length

Line graph A graph using line segments to show values (usually over a period of time)

Lowest terms A fraction is in lowest terms when numerator and denominator do not contain a common factor other than 1

Mathematical sentence A statement of equality or inequality which may be a true statement, a false statement, or a conditional statement

Means The second and third terms of a proportion

Measurement The process of associating number with some property of an object

Metric system The system of weights and measures based on the meter (39.37 inches) and the gram (0.0022 pounds approximately)

Minuend The number from which another is to be subtracted

Mixed numeral The indicated sum of a natural number and a fractional number

Multiplicative inverse If the product of two numbers is 1, each is the multiplicative inverse, or reciprocal, of the other.

Mutually exclusive events Sets of events having no elements in common

Natural numbers The set of numbers used for counting, $\{1, 2, 3, \ldots\}$

Negative integers The set of numbers, $\{\ldots, {}^{-}3, {}^{-}2, {}^{-}1\}$

Negative numbers Numbers having values less than zero

Numeration system A system for writing numerals

Number line A line used in graphing points associated with numbers

Numeral A name for a number

Numerator The number indicated above the bar in a fraction indicating how many of the equal parts are being considered

Obtuse angle An angle whose measure is greater than 90° but less than 180°

Obtuse triangle A triangle which contains an obtuse angle

Octagon An eight-sided polygon

Open sentence A mathematical sentence which is neither true nor false; a conditional statement; i.e., $x + 2 = 6$

Parallel lines Lines in the same plane with no point in common

Pentagon A five-sided polygon

Perimeter The distance around a polygon

Permutation Each different ordered arrangement of elements in a set

Perpendicular lines Lines which meet or intersect to form right angles

Pi The number which is the ratio of the circumference to the length of a diameter of a circle

Pictograph A graph in which quantities are represented by symbols; each symbol representing a stated number of units

Plane A set of points making up a flat surface extending indefinitely in two dimensions

Point The unit element in geometry

Polygon A closed plane figure, each of whose sides is a segment

Positive integers The set of natural numbers

Positive numbers Numbers whose value is greater than zero

Prime factorization Factorization utilizing only prime numbers

Prime number A number with no factors other than itself and 1

Prism A solid figure, each of whose faces is a parallelogram and whose bases are congruent and parallel

Probability The chance that an event will or will not occur

Product The result obtained from multiplying factors

Proper fraction A fraction whose denominator is greater than its numerator

Proportion A statement of equality between two ratios

Protractor An instrument for measuring an angle in degrees

Quadrilateral A four-sided polygon

Quotient The number of times the divisor is contained in the dividend, or the second factor found when the product and one factor are given

Radius A segment, one of whose endpoints is the center of the circle and whose other endpoint is on the circle

Ratio A comparison, by division, between two numbers which may be expressed as a fraction, a decimal, or a per cent

Rational numbers The set of numbers that can be expressed as the ratio of two integers, $\frac{a}{b}$, $b \neq 0$

Ray A subset of a line formed by any point of the line and all the points on one side of the chosen point

Rectangle A quadrilateral whose angles are right angles

Regular polygon A polygon the measures of whose angles are the same and the measures of whose sides are the same

Relatively prime Two numbers are relatively prime if there is no prime number that is a factor of both.

Remainder That which is left (less than the divisor) after division is performed, or the difference when the subtrahend is subtracted from the minuend

Right angle An angle whose measure is 90°

Right triangle A triangle with one right angle

Sample space The set of elements constituting the ways in which an event can happen

Scale drawing A drawing in which the measure of each of the parts has a stated ratio to the measure of the object it represents

Scalene triangle A triangle in which the measures of no two sides are the same

Sector A part of a circular region bounded by an arc and two radii

Segment A subset of a line formed by any two points of the line and all the points between them

Semicircle Half of a circle

Set Any well-defined collection

Signed numbers Numbers whose values relative to zero are indicated by + and −

Similar figures Figures that have the same shape but not necessarily the same size

Square A regular quadrilateral

Subset A set, all of whose members are also members of a given set

Subtrahend The number to be subtracted from the minuend

Sum The result from addition

Terms The numerator and denominator of a fraction; the means and extremes in a proportion; the symbols separated by + or − signs or verbs in a mathematical sentence

Tolerance The greatest discrepancy permitted from the specified dimension

Trapezoid A quadrilateral with one pair of parallel sides

Triangle A three-sided polygon

Unit of measurement A stated amount or quantity adopted for purposes of comparison

Value of a digit The number named by the digit multiplied by the value of its place in a numeral

Variable A symbol used to hold the place of a numeral

Vertex The common endpoint of two rays forming an angle

Volume The number of cubic units contained in a three-dimensional figure

Whole numbers The set of numbers consisting of the natural numbers and zero, $\{0, 1, 2, \ldots\}$

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